# Brief Announcement: A Log*-Time Local MDS Approximation Scheme for Bounded Genus Graphs*

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**Abstract.** This paper shows that the results by Czygrinow et al. (DISC 2008) and Amiri et al. (PODC 2016) can be combined to obtain a $O(\log^* n)$-time local and deterministic approximation scheme for Minimum Dominating Sets on bounded genus graphs.

## 1 Local MDS Approximation Scheme

It is well-known that fundamental graph problems such as the Minimum Dominating Set (MDS) problem cannot be solved efficiently by distributed algorithms on general graphs. However, over the last years, researchers have found several very fast distributed algorithms for sparse families of networks, such as constant-degree graphs and planar graphs.

This paper presents a deterministic $O(\log^* n)$-time MDS $(1+\epsilon)$-factor approximation algorithm for a more general graph family: graphs of constant genus. The algorithm relies on: (1) a slight modification of the clusting algorithm for planar graphs presented by Czygrinow et al. [2], and (2) the recent constant approximation result by Amiri et al. [1] for MDS on graphs of bounded genus. Due to space constraints, we refer the reader to the prior work for more background.

We suppose familiarity with basic graph theory and graphs on surfaces [4]. We consider simple finite undirected graphs unless stated explicitly otherwise. We denote the set of all integers by $\mathbb{N}$. For a graph $G = (V, E)$, we write $E(G)$ resp. $V(G)$ to denote the edge set resp. vertex set of graph $G$. For a *weighted* graph $G$, we define an edge weight function as $w : E(G) \to \mathbb{N}$. For a sub-graph $S \subseteq G$, we write $W(S)$ for $\Sigma_{e\in E(S)} w(e)$, and call it the total edge weight of $S$. We contract an edge $\{u, v\}$ by identifying its two ends, creating a new vertex $uv$, but keeping all edges (except for parallel edges and loops). Additionally, if the graph is weighted and $\{u, x\}, \{v, x\} \in E(G)$, we set the edge weight of $\{uv, x\}$ to $w(uv, x) := w(u, x) + w(v, x)$. Let $S \subseteq V(G)$, we denote by $G[S]$ an induced subgraph of $G$ on vertices of $S$. The *degeneracy* of a graph $G$ is the least number $d$ for which every induced subgraph of $G$ has degree at most $d$.

We need the following lemma for the sake of completeness.

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**Lemma 1.** *Let $\mathcal{G}$ be a class of graphs of genus at most $g$. Then the degeneracy of every graph $G \in \mathcal{G}$ is in $O(\sqrt{g})$.*

*Proof.* We prove the lemma for graphs with orientable genus $g$; an analogous argument works for graphs of non-orientable genus $g$. Let $G \in \mathcal{G}$ with genus at most $g$, and suppose the degeneracy of $G$ is $c$. We prove that $c \in O(\sqrt{g})$. Let us denote by $v, e$ the number of vertices and edges of $G$, respectively. By the Euler formula, we have: $e \leq 3 \cdot v + 6g - 6$ [1]. On the other hand, by definition of the degeneracy, every vertex in $G$ has degree at least $c$, so $\frac{c \cdot v}{2} \leq 3v + 6g - 6 \Rightarrow c \leq \frac{12g-12}{v} + 6$ (1). To find the maximum value of $c$ for a fixed genus, we must minimise $v$. A complete graph on $v$ vertices has genus at most $v^2/12$ [4], therefore by plugging it into (1), we obtain that $c \leq \sqrt{12g} + 6$.

**Definition 1 (Pseudo-Forest [2]).** *A* pseudo-forest *is a directed graph in which every vertex has an out-degree at most* 1*.*

For a directed graph $G$, if we ignore the edge directions, we write $\bar{G}$.

Different variations of the first part of the following lemma have already been proved in the literature. However, to be able to provide exact numbers and for completeness, we include a proof here. Let $G$ be a graph and let $\mathcal{F}$ be a family of forests such that for all $F \in \mathcal{F}$, we have $F \subseteq G$. We say that $\mathcal{F}$ is a *forest cover of* $G$, if for every edge $e \in E(G)$, there is a forest $F \in \mathcal{F}$ such that $e \in E(F)$.

**Lemma 2.** *There is a constant $c_1$ such that for an edge weighted graph $G$ of genus $g$, we can find, in two communication rounds, a pseudo-forest $F$ such that $\bar{F}$ is a spanning sub-graph of $G$ and $W(\bar{F}) \geq \frac{W(G)}{c_1 \cdot \sqrt{g}}$.*

*Proof.* By Lemma 1, the degeneracy of a graph $G$ of genus $g$ is in $O(\sqrt{g})$. The degeneracy is within factor two of the arboricity [3], and the arboricity equals the size of at least a forest cover $\mathcal{F}$ of $G$. Therefore, there is a constant $c'_1$ such that $|\mathcal{F}| \leq c'_1 \cdot \sqrt{g}$. Hence, there is a forest $F_1 \in \mathcal{F}$ such that $W(F_1) \geq W(G)/(c'_1 \cdot \sqrt{g})$. Similarly to the proof of Fact 1 in [2], for a vertex $v$, we choose an edge $\{v, u\}$ of largest weight, and direct it from $v$ to $u$. If we happen to choose an edge $\{v, u\}$ for both vertices $u$ and $v$, we direct it from $v$ to $u$, using the larger identifier as a tie breaker. This algorithm creates a pseudo-forest $F$. $\bar{F}$ is a spanning sub-graph of $G$ and it has a total edge weight of at least half of $W(F_1)$, so $W(\bar{F}) \geq W(G)/(2 \cdot c'_1 \cdot \sqrt{g})$. We set $c_1 = 2 \cdot c'_1$. Note that we found $F$ in two rounds. □

**Lemma 3.** *There is a local algorithm which takes an $0 < \epsilon < 1$ and an edge-weighted graph $G$ of genus at most $g$ as input, runs in $O(\log^* n + 1/\epsilon \cdot \sqrt{g})$ communication rounds and returns a set of clusters $C_1, \ldots, C_l$ partitioning $G$, such that, each cluster has a constant diameter. Moreover, if we contract each $C_i$ to a single vertex to obtain a graph $H$, then $W(H) \leq \epsilon \cdot W(G)$.*

*Proof.* Let $t := 4 \cdot c_1 \cdot \sqrt{g}$. By applying the HeavyStars algorithm from [2] on the pseudo forest provided in the proof of Lemma 2, we obtain stars of weight $\frac{|E(G)|}{t}$.

We also run the algorithm Clustering provided in [2], but we set the number of iterations in the algorithm to $\log(\frac{1}{\epsilon})/\log(\frac{t}{t-1})$; the rest of the algorithm is left unchanged. A similar line of proof for the original algorithm, proves the claim of the lemma. Just note that $\log(\frac{x}{x-1}) \geq 1/x$ for $x > 1$. $\square$

**Theorem 1.** *Given a $0 < \delta < 1$ and a graph $G$ of bounded genus, the Minimum Dominating Set can be approximated in $O(\log^* |G|)$ time within a factor of $1+\delta$.*

*Proof.* Suppose $OPT$ is the optimal dominating set of $G$. By [1], we can find a dominating set $D$ of $G$ such that for some constant $c$, we have $|D| \leq c \cdot g \cdot |OPT|$. This can be done in a constant number of communication rounds. For a vertex $v \in G$, we denote the neighbours of $v$ in $G$ by $N[v]$ i.e., $N[v] = v \cup \{u \in V(G) \mid \{u, v\} \in E(G)\}$. Suppose $|D| = t$.

Let us order the vertices of $D$ arbitrarily, and suppose $d_1, \ldots, d_t$ is such an ordering. Create a partition $(V_1, \ldots, V_t)$ of $V(G)$ such that $V_i = \{v \in N[d_i] \mid v \in (G - D - \bigcup_{j<i} V_j)\} \cup \{d_i\}$. We next contract each $V_i$ to a single vertex $v_i$ to obtain a graph $H$. We assign an edge weight to $H$, i.e., for all $e \in E(H)$, we set $w(e) := 1$. It is clear that $W(H) = |E(H)|$. $H$ has genus at most $g$ and it has at most $3|D| + 6g - 6$ edges (see Lemma 4 of [1]). Set $\epsilon = \delta/((6 + 12g) \cdot c \cdot g)$. When we apply the algorithm in Lemma 3, it finds clusters $C_1, \ldots, C_l$ such that the total edge weights between clusters amount to at most $\epsilon \cdot |E(H)|$ . Note that as $\epsilon \in \Omega(1/g^2)$, the algorithm uses $O(\log^* |G| \cdot 1/\epsilon \cdot \sqrt{g}) = O(\log^* |G| \cdot g^2 \sqrt{g})$ communication rounds.

For a cluster $C_j$, suppose $V(C_j) = \{v_{j_1}, \ldots, v_{j_k}\}$, and let $U_j$ be an induced subgraph of $G$ on vertices of a subgraph $X = \bigcup_{i=1,\ldots,k} V_{j_i}$, i.e $U_j := G[X]$. We find the optimum dominating set $S_i$ in each $U_j$. Moreover, we know that each $C_i$ had a constant diameter therefore, each $U_j$ will have a constant diameter. Hence, finding an optimum dominating set within each $U_i$ can be done in a constant number of communication rounds. Now take a dominating set $S = \bigcup S_i$. First of all, it is clear that $S$ is a dominating set of $G$. To prove the upper bound, let $D^*$ be a set of vertices of $D$ which have a neighbour in other clusters, i.e., $D^* = \{w \in D \mid \text{if } w \in U_i \text{ then } \exists j \neq i \text{ and } \exists x \in U_j \text{ such that } \{w, x\} \in E(G)\}$. By the Clustering algorithm and the above counting, we have $|D^*| \leq 2\epsilon|E(H)| \leq 2\epsilon(3|D| + 6g - 6) \leq 2\epsilon \cdot c \cdot g \cdot (3|D| + 6g) \leq \delta|OPT|$. On the other hand, we know that $|S| \leq |OPT \cup D^*| \leq (1 + \delta)|OPT|$. $\square$

## References


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