Evacuating Two Robots from Two Unknown Exits on the Perimeter of a Disk with Wireless Communication

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ABSTRACT

The evacuation of mobile robots is an interesting emerging application in distributed computing. This paper considers the fundamental problem of how to evacuate two robots from a unit disk. The robots, initially located at the center of the disk, need to exit the disk through two unknown exits, at known distance d from each other, located at the perimeter of the disk. The robots can coordinate when exploring the disk, using wireless communication. The objective is to minimize the evacuation time, i.e., the time until the last robot exits the disk. We consider two different model variants, where exits can either be labeled or unlabeled. We complement our analysis with simulations.

CCS CONCEPTS

• **Theory of computation** → **Design and analysis of algorithms**; **Distributed algorithms**;

KEYWORDS

Mobile Robots, Evacuation, Distributed Algorithm

1 INTRODUCTION

Searching is an inherent problem in computer science. Being an intriguing field with a long history, a plethora of research has been conducted on search problems using a multitude of

models, including probabilistic search models [11], cops and robbers models [2], search problems in groups [1], classical pursuit and evasion [4, 9], to just name a few. In these papers, the goal is typically to find an object located in a specific domain.

This paper is situated in the context of emerging robot evacuation problems in which robots collaboratively search for exits. In particular, we consider the fundamental problem of evacuating two robots through two unknown exits located on the perimeter of a unit disk. Unlike many traditional search problems, we aim to minimize the time needed for the *last robot* to exit the disk. While initially the robots have no knowledge about the positions of exits, they have some information about the distribution of exits. In particular, we assume that the robots know that the exits are at distance d from each other, for some parameter d. Moreover, the robots can cooperate with each other to locate the exits, using wireless communication.

1.1 Related Works

Search problems for robots are an emerging field and have recently received quite some attention. In particular, search problems on a disk have been introduced by Czyzowicz et al. [6]: the authors distinguish between two different communication models, *wireless* and *face-to-face*. In the wireless model, it is assumed that robots can communicate at any point in time. In particular, once one robot finds an exit, it can immediately communicate its location to the other robot who could then use the same exit (which however may not be the optimal strategy for the second robot at this point). The paper presents algorithms providing optimal worst-case evacuation times (namely $1 + \frac{2\pi}{3} + \sqrt{3} \approx 4.826$) for two robots in the wireless model. In the face-to-face communication model, robots can only communicate if they are located at the same point: Czyzowicz et al. [6] provide an upper bound of 5.740 as well as a lower bound of 5.199, for a single exit with two robots. Later, Czyzowicz et al. [7] improved the bounds for two robots with one exit in a disk, by proposing linear and triangular detours at the worst case positions. More specifically, they improved the upper bound to 5.628 and the lower bound to $3 + \frac{\pi}{6} + \sqrt{3} \approx 5.255$. Recently, Brandt et al. [3] further improved the upper bound to 5.625 using linear detours which do not stipulate the robots to meet. In [5], the robots can travel only on the perimeter of the circle and use

wireless communication. For this model, the authors provide upper and lower bounds for different distributions of exits with multiple exits. In [8], robots with different speeds are considered, in the wireless communication model.

1.2 Model and Preliminaries

We in this paper consider evacuation problems for two robots R_1 and R_2 from a unit disk. The disk contains two doors, located on the perimeter of the disk, which the robots can use to exit the disk (both robots can but do not have to use the same exit). Initially, the robots are situated at the center of the disk. We assume that the only information the robots have about the exits is the distance d between them: the length of the smaller arc between them. Accordingly, the length of a chord for such an arc is $2\sin(d/2)$. In the following, we let $\widehat{AB}$ denote the smaller arc over the perimeter joining two points A and B: $\widehat{AB}$ denotes the arc starting from A and moving in the clockwise direction until B. So $\widehat{AB}$ and $\widehat{BA}$ together cover the entire perimeter of the disk. We will denote by $\overline{AB}$ the line segment joining points A and B. We use $|\widehat{AB}|$ and $|\overline{AB}|$ to denote the length of the arc and line segment, respectively.

The initial separation between the robots ζ is a known value. Both robots travel at uniform speed, i.e., 1 unit distance per unit time and can move anywhere within the disk. The robots can see the exits only when they reach the corresponding location. However, they can learn about the coordinates of the location from the other robot using *wireless* communication. Communication is reliable and message propagation delay is ignored. We differentiate between *labeled* and *unlabeled* exits. Labeled exits have identities and are ordered, either clockwise or counter-clockwise. This means when a robot encounters an exit, in the labeled model variant, it can determine the location of the other exit at distance d; in the unlabeled model, there would still be two options left at this point. The robots can exit via the same or different exits.

1.3 Our Contribution

We present distributed evacuation algorithms for two robots to evacuate via the exits situated on the perimeter of a unit disk, starting from the center. We study unlabeled and labeled problem variants.

- In Section 2, we propose and analyze generic evacuation algorithms which are parametrized with a distance $\zeta \le d$ at which robots hit the perimeter the first time.
- In Section 3, we study the algorithms using simulations for both labeled and unlabeled exits, shedding light on how the worst-case evacuation time varies with d. Our algorithm achieves a worst case evacuation time 4.826 for the $d = 0$ scenario presented in [6].

2 ALGORITHM AND ANALYSIS

2.1 Problem Statement and Algorithm

PROBLEM 1. *Two robots are placed at the center of a unit disk containing two exits on the perimeter with d as the minor arc length between exits. Given d as an input, the objective is to minimize the worst case exit time for all the robots to evacuate the disk, where exits are labeled or unlabeled.*

Initially, the robots simply start moving from the center O towards the boundary of the disk, along the radius. The robots may move in different directions, however. Our algorithm is evoked once a robot encounters an exit or receives a message.

Assume the robots hit the perimeter at two different points B and C, and we define $|\widehat{BC}| = \zeta$ as shown in Fig 1. Moreover, we define $\overrightarrow{OA}$ to be the positive x-axis. For simplicity, we define A as the midpoint of $\widehat{BC}$. So $|\widehat{BA}| = |\widehat{AC}| = \zeta/2$. If $\zeta = 0$, then B and C coincide at A. Let robot R_1 move in the counter-clockwise direction and robot R_2 in the clockwise direction. Suppose two exits are located at E_1 and E_2.

Without loss of generality, let us assume that R_1 finds E_1 at X before R_2, unless both find the exits simultaneously. Now, R_1 sends a message to R_2 that it found the exit. Given the two robots have the same velocity, R_2 can find out the location of the exit E_1 based on the position of R_1. Say $|\widehat{XB}| = |\widehat{CD}| = x$. Since the robots know d, the distance over the arc between the exits, in the unlabeled setting, R_2 can predict two probable positions for the exit at E_2' and E_2''. Let E_2' be the closest exit in clockwise direction and E_2'' is the closest exit in counter-clockwise direction from E_1. This implies $|\widehat{E_1 E_2'}| = |\widehat{E_2'' E_1}| = d$.

The algorithm for evacuation follows two simple subroutines when encountering an exit and receiving a message. If R_1 encounters an exit E_1 at X, then it sends a message to R_2. Once R_2 receives a message, it can determine the location of R_1 according to the time it received the message and designate that location as a location of exit. Then R_2 can compute possible locations of exits at a distance d from that exit. Then R_2 determines its path by choosing either the definite exit or two probable exit locations for unlabeled exits. For labeled exits, R_1 can send a one bit message to relay direction of E_2 with respect to E_1, i.e., it sends 1 for clockwise and 0 for counter-clockwise. Then R_2 can choose the closest exit. Then it leaves via that exit.

2.2 Labeled Exits

Consider the following functions as per the cases. If R_2 evacuates via E_1, then the time required is

$$z_1 = 1 + x + 2\sin(x + \zeta/2) \qquad (1)$$

If R_2 evacuates via E_2', then the time required is

$$z_2 = 1 + x + 2\sin(x + (\zeta - d)/2) \qquad (2)$$

If R_2 evacuates via E_2'', then the time required is

$$z_3 = 1 + x + 2\sin(x + (\zeta + d)/2) \qquad (3)$$

The domain of x for each function is different as presented in the different cases. There can be the following situations if E_1 is located at X (ref. Fig. 1). As we consider E_1 is the exit encountered first by R_1, so E_2 cannot lie on $\widehat{XB}$ and $\widehat{CD}$.

Case 1: E_2' is E_2
- E_2' lies on $\widehat{DX}$, this leads to $|\widehat{XE_2'}| > |\widehat{XD}|$, i.e., $d > 2x + \zeta$.
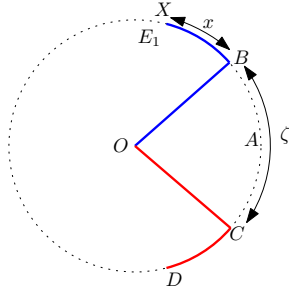
Figure 1: R_1 encounters E_1 at X

- E_2' lies on $\widehat{BC}$, this leads to $|\widehat{XC}| > |\widehat{XE_2'}| > |\widehat{XB}|$,
  i.e., $x + \zeta > d > x$.
  So, the domain of x is $[0, (d-\zeta)/2] \bigcup [d-\zeta, d]$. The time for evacuation is $\min(z_1, z_2)$.

Case 2: E_2'' is E_2

- E_2'' lies on $\widehat{DX}$, this leads to $|\widehat{E_2''X}| < |\widehat{DX}|$,
  i.e., $d < 2\pi - (2x + \zeta)$.
- E_2'' lies on $\widehat{BC}$, this leads to $|\widehat{XB}| > |\widehat{XE_2''}| > |\widehat{CX}|$,
  i.e., $2\pi - x > d > 2\pi - x + \zeta$.
  So, the domain of x is $[0, \pi - (d + \zeta)/2] \bigcup [2\pi - (d + \zeta), \min(\pi, 2\pi - d)]$. The time for evacuation is $\min(z_1, z_3)$.
  The worst case time for evacuating the disk is

$$\inf_{\zeta}\left(\sup_{x}\left(\min(z_1, z_2), \min(z_1, z_3)\right)\right) \qquad (4)$$

2.3 Unlabeled exits

For unlabeled exits, the cases are described as following.

Case 1: Both E_2' and E_2'' are unexplored
In this case, R_2 can move towards the definite exit at X or it can go to the two probable exit positions E_2' and E_2''. It chooses the minimum of the two. The linear distance between the two probable exits, z_4, is

$$z_4 = 2\sin(\pi - d) \qquad (5)$$

There can be three different situations in this case.
- E_2' and E_2'' lies on $\widehat{DX}$,
  i.e., $2x + \zeta \leq d$.
- E_2' is on $\widehat{DX}$ and E_2'' is on $\widehat{BC}$,
  i.e., $x \leq d < x + \zeta$ and $d + 2x + \zeta < 2\pi$
- E_2' and E_2'' are on $\widehat{BC}$,
  i.e., $x \leq d < x + \zeta$ and $d + x + \zeta > 2\pi$
The time for evacuation is $\min(z_1, z_2 + z_4, z_3 + z_4)$.

Case 2: E_2' is unexplored,
i.e., $x < d$ and $x + \zeta + d < 2\pi < 2x + \zeta + d$
In this case, as E_2'' is already explored, so there is definitely an exit at E_2'. So the time for evacuation is $\min(z_1, z_3)$.

Case 3: E_2'' is unexplored,
In this case, as E_2' is already explored, so there is definitely an exit at E_2''. So the time for evacuation is $\min(z_1, z_2)$. This case happens in the following two situations.

- E_2' is on $\widehat{CD}$ and E_2'' lies on $\widehat{DX}$,
  i.e., $x + \zeta \leq d < 2x + \zeta$ and $d + 2x + \zeta < 2\pi$.
- E_2' is on $\widehat{XB}$ and E_2'' lies on $\widehat{DX}$,
  i.e., $d < x$ and $d + 2x + \zeta < 2\pi$

THEOREM 2.1. *As soon as one robot finds an exit, the other robot can determine the best location to exit for itself.*

PROOF. If both the robots find the exits simultaneously, then they exchange messages and an agreement is achieved. Both the robots exit via their respective exits.

If one robot finds the exit, then it sends a message to the other robot. When the second robots receives the message, it can determine the location of one exit and two probable exit locations. Then it determines the path of evacuation by computing the least amount of travel distance. Hence an agreement is achieved. □

The cases mentioned above are applicable for $0 \leq \zeta \leq d$. If $d < \zeta$, then the worst case arises in the situation where both the exits lie on $\widehat{BC}$, the unexplored part of the perimeter as shown in Fig. 1. So, the robots have to go back the unexplored part of the perimeter, when they do not find the exits. This increases the evacuation time compared to the algorithm for $\zeta \leq d$. This case can be considered independent of the series of cases described before this, since the condition for this case is independent of x. In this special case the time of evacuation is always greater than $\widehat{DB} + \overline{DB}$, i.e., $\pi - \zeta/2 + 2\sin(\pi - \zeta/2)$.

3 SIMULATION RESULTS

We conducted simulations to study the evacuation times for different $d \in [0, \pi]$ and different ζ values, in particular $\zeta = 0$, $\zeta = d/2$ and $\zeta = d$ (for unlabeled exits) and $\zeta \in [0, d]$ (for labeled exits). Fig. 2 plots the worst case time for evacuation versus the distance between two exits, d. For each value of d and ζ, we considered all possible position of exits with a 0.001 step size for x and took the maximum over it to determine the worst case time. Then we varied the d values with a 0.001 step size with corresponding ζ value. It can be observed from Fig. 2 that the worst case evacuation time is less for $\zeta = d$ for most of the values of d. Since this is a search problem, reducing the search space can be an effective method to reduce the worst case evacuation time. As we have two exits and we know the distance between them, we can easily remove an arc length equal to d from the perimeter because two exits cannot lie within a d distance arc. From Fig. 2, it is clear that $\zeta = d$ performs better compared to $\zeta = 0$ and $\zeta = d/2$. Even, $\zeta = 0$ performs better than $\zeta = d/2$ for $d > 1.21$. It can be observed from the figure that, it is not strictly monotonous for $\zeta = 0$ or $\zeta = d$. The reason for this is that there is a transition between cases when there is a local minimum or local maximum.

As per Fig. 2, for $\zeta = 0$, it is monotonic until $2\pi/3$. For $d \leq 2\pi/3$, it follows Case 2 with z_3 as the worst case time. But for $d > 2\pi/3$, it changes to Case 1 and loses monotonicity. As per Fig. 2, for $\zeta = d$, observe that there is a local minimum at $d = 0.93$. The minimum marks the end of Case 2. For $0.93 < d \leq 2\pi/3$, both the probable exits are unexplored and
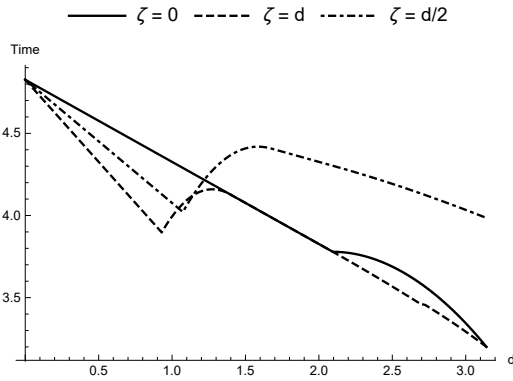
Figure 2: Comparision of Algorithms for different ζ values for wireless communication with unlabeled exit locations

so the evacuation time increases and then decreases. In the range, where the evacuation time increases until it reaches the local maximum at $d = 1.385$, the unexplored exit E_1' lies on $\widehat{BC}$ very close to B. For $2\pi/3 < d \le 2.69$, the evacuation time monotonically decreases according to Case 3. For $d > 2.69$, Case 1 becomes the worst case.
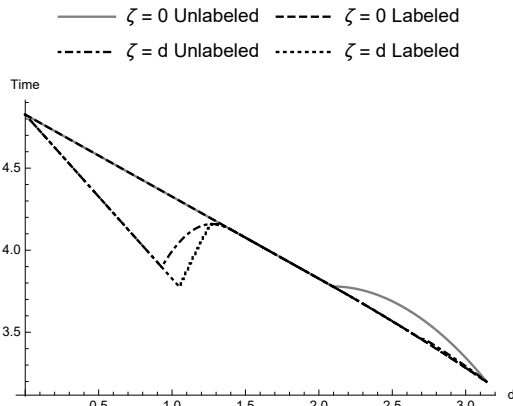

Figure 3: Comparision between evacuation time of Unlabeled and Labeled exit locations for $\zeta = 0$ and $\zeta = d$

In Fig. 3, we plot the evacuation time for a labeled exit location and unlabeled exit locations for $\zeta = 0$ and $\zeta = d$. It can be easily observed that the labeled exit time is strictly monotonic for $\zeta = 0$. For $\zeta = d$, the evacuation time for labeled exits falls closely with unlabeled exits. In Fig. 4, we plot minimum evacuation time for labeled exit locations over values of $\zeta \in [0, d]$.

4 CONCLUSION

In this paper, we address the evacuation problem for two robots in a unit disk with two exits located at arbitrary points on the perimeter. We have considered both unlabeled and labeled exits in the wireless communication model. More details and variations of the problem are available at [10].
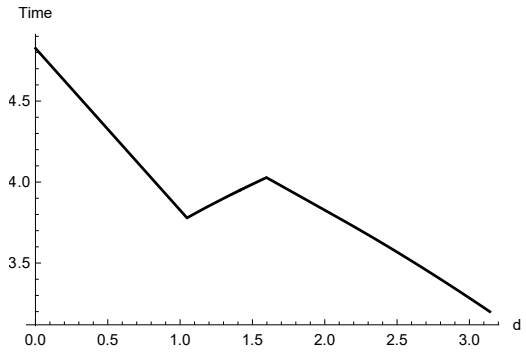

Figure 4: The minimum evacuation time over $\zeta \in [0, d]$ for labeled exits

The previous papers consider only a single exit with $\zeta = 0$, and the evacuation time for our algorithm is also 4.826 for $d = 0$ in the same model. We believe that our paper opens interesting directions for future research. For example, it is still open to find out a function which describes the relation between ζ and d in the same model. We think that this is very close to the optimal solution for the wireless communication model.

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