# A Local $O(1)$-Approximation for Dominating Sets on Bounded Genus Graphs

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# Plan

# Local Model

Linial 1992:

1. Input is a graph $G = (V, E)$.
2. Every vertex is an oracle which can do every computation in one unit of time.
3. Computation is synchronous and reliable.
4. In each communication round a vertex may pass arbitrary large messages to some of its neighbors.
5. Distributed complexity defined as number of communication rounds.

# Complexity of Dominating Set Problem in Classical and Local Model

1. Classical model : PTAS for planar graphs and graphs of excluded minor [Baker 1994,Grohe 2003].
2. Finding dominating sets locally is hard [Kuhn et al 2010].
3. No deterministic local algorithm with approximation factor $7-\epsilon$ for planar graphs in $O(1)$ communication rounds [Hilke et al. 2014].
4. $O(1)$-approximation in $O(1)$-communication rounds for planar graphs [Lenzen et al. 2013].

# Plan

1. *Star* : $K_{1,n}$, center of star has degree $n$.
2. *Star Contraction* : Contracting disjoint stars to their center.
3. $H$ is a *depth*-1 *minor* of a graph $G$, if it is a star contraction of a subgraph of $G$.
4. The edge density of $G$ is $\epsilon(G) = |E(G)|/|V(G)|$.

**Lemma 1 (Theorem 4.4.7 Mohar and Tomassen, Graph on Surfaces)**

*If $G$ is of genus $g$, then $G$ excludes $K_{4g+3,3}$ as minor, and thus as a depth-1 minor.*

**Lemma 2**

*If $G$ is of genus $g$, then $G$ contains at most $g$ disjoint copies of $K_{3,3}$ as minors, and in particular as depth-1 minors.*

**Theorem 3**

*There is a local $O(g)$-approximation algorithm which approximates dominating set in the class of graphs of genus at most $g$. Furthermore the number of communication rounds is in $O(g)$.*

# Plan

# The Algorithm

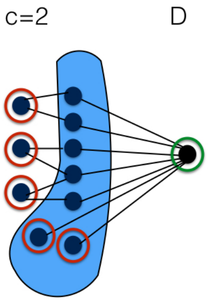
The algorithm consists of two phases.

1. In Phase 1, take a set $D$ in dominator set: Set of *big* vertices.
2. In Phase 2, find a dominator set for $V - N[D]$.

# Phase 1

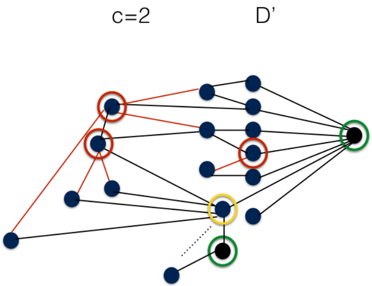
$c$ : the edge density of depth-1 minors of $G$

Choose a set $D \subseteq V(G)$ where for every $v \in D$, $N[v]$ cannot be covered by $2c$ vertices in $V(G) - \{v\}$.



# Phase 2 : Finding Dominators of $V - N[D]$

1. $D' := \emptyset$
2. For all $v \in V - D$ : $\tilde{d}(v) := |N(v) - D|$.
3. $v \in V - N[D]$ : $D' := D' \cup \{z\}$ such that $\tilde{d}(z) = \max_{u \in N[v]} \tilde{d}(u)$.



# Plan

# Proof at a glance

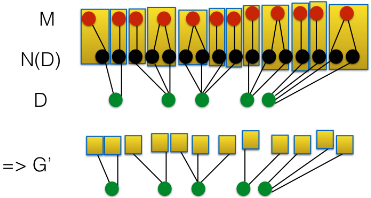
1. Number of vertices in $D$ is small by a simple counting argument.
2. $D'$ is small: We use the fact that the graph is sparse and has no $K_{3,t}$.

# Size of the Set $D$ in Uniformly Sparse Graphs is Small

**Lemma 4**

$|D| \leq (2c+1) \cdot |M|$.

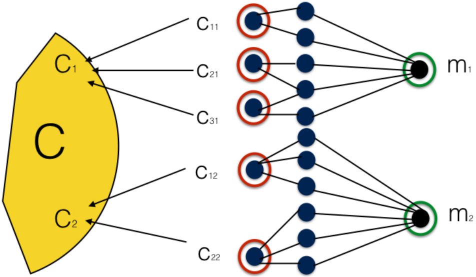
Suppose $D \cap M = \emptyset$.
$E(G') \geq (2c+1) \cdot |D|, E(G') \leq 2c \cdot (|M| + |D|) \rightarrow |D| \leq 2c \cdot |M|$



# Assumption:

1. $G$ excludes $K_{3,t}$ as depth one minor.
2. Every depth-1 minor of $G$ has constant density (at most $c$).
3. $M = \{m_1, \dots, m_{|M|}\}$ is the optimal dominating set of $G$.
4. $C_i \subset V - \{m_i\}$ is a set of vertices of size at most $2c$ which dominates $N[m_i]$ for $m_i \in M - D$ (otherwise empty set).
5. $C = \bigcup_{i=1}^{|M|} C_i$.

# Illustration of the C Set



# Star graph $S^G$ of $G$

$S^G$ : disjoint union of $|M|$ star subgraphs $S_1, \ldots, S_{|M|}$ of $G$ with centers $m_1, \ldots, m_{|M|}$ for $m_i \in M$.

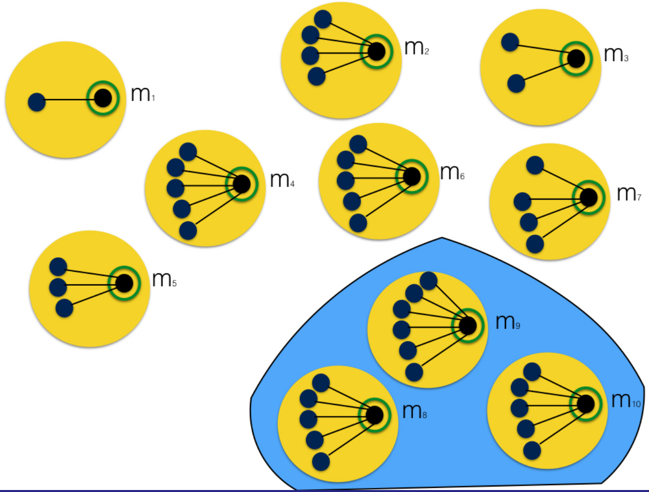
1. Start with $S^G = (V(G), \emptyset)$.
2. Add minimum number of edges from $E(G)$ to $E(S^G)$, such that $m_1, \ldots, m_{|M|}$ dominates all vertices in $S^G$.

# Star Contraction Graph $S_G^o$

1. Start with $S_G^o = S^G$.
2. Add every edge $\{u, v\} \in E(G)$ to $E(S_G^o)$ if $u, v \notin M$ and $u \in V(G) - D$.
3. Contract all stars $S_1, \ldots, S_{|M|}$.

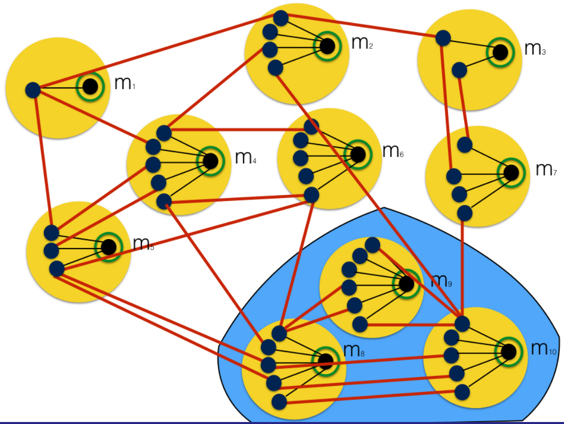
Number of vertices in $S_G^o$ is $|M|$ and number of edges is at most $c|M|$.

# Illustration of Star Graph of $G$

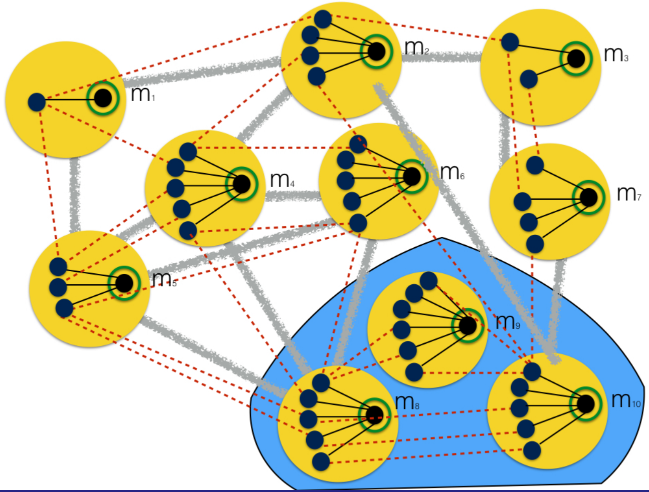


# Illustration of Star Contraction Graph of $G$

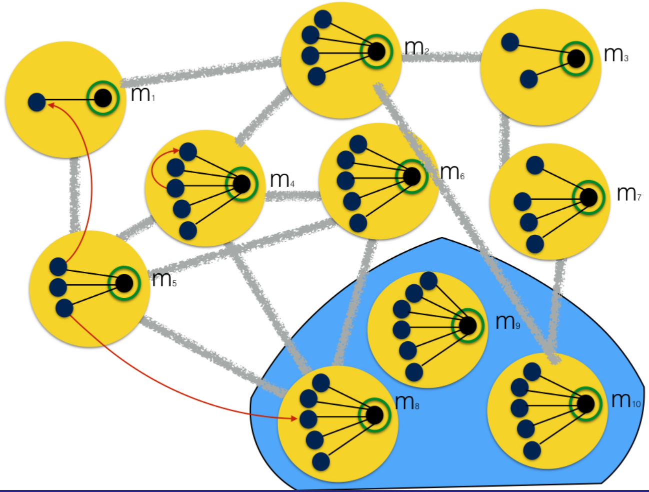
Edge between two stars could be an edge from a dominator vertex to a dominated vertex.



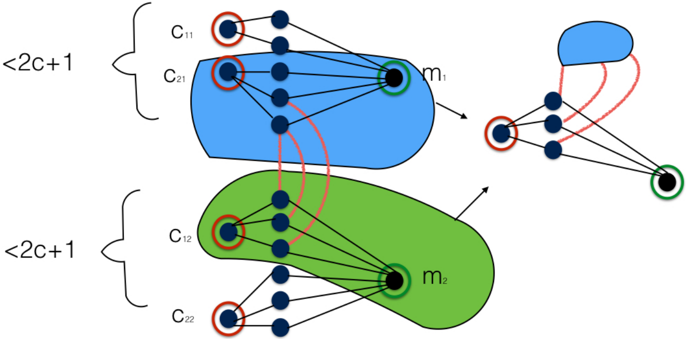
# Illustration of Star Contraction Graph of $G$



# Star Contraction Graph and Different Kind of dominations

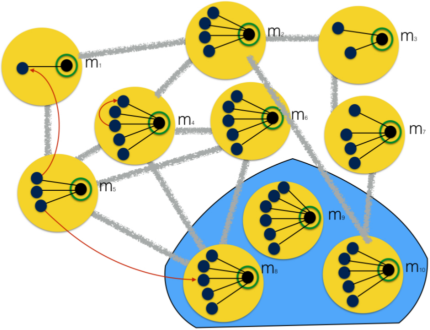


# Each $e \in E(S_G^o)$ corresponds to at most $2c(t-1)$ dominator edges in $G$



# Total number of edges between stars is in $O(c^2 \cdot t \cdot |M|)$

1. Each gray edge corresponds to $O(c \cdot t)$ edges in $G$.
2. Total number of gray edges is $O(c \cdot |M|)$



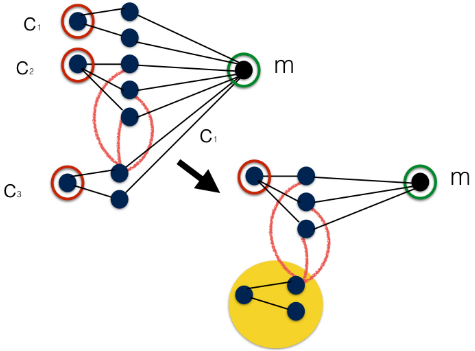
# What additional dominators exist?

1. Similarly we can count number of dominator edges between $N[D]$ and the $V - N[D]$.
2. There can be a dominator edge which connects two vertices in a star.
3. We have to count those dominator vertices which do not have any edge to other stars.

# Counting Number of Dominator Edges in the Star

1. Suppose every vertex inside a star has at most $O(c)$ edges to $M$.
2. The degree of the vertex cannot be bigger than $O(c \cdot t)$ otherwise we find $K_{3,t}$.
3. So the corresponding star is not big (at most $O(c^2 \cdot t)$).
4. Total number of stars is $|M|$ so there are at most $O(c^2 \cdot t|M|)$ such dominator edges (resp. dominator vertices).

# Total Number of Edges Inside One Star is Small Otherwise There is a $K_{3,t}$.



**Lemma 5**

*There are at most $|M|$ vertices which have more than $2c$ edges to $M$.*

# Proof of Theorem 3

*Proof.* If $G$ excludes $K_{3,t}$ and has edge density at most $c$:

1. There are $|M|$ stars.
2. There are $O(c^2 \cdot t|M|)$ dominator edges between stars.
3. There are $O(c|M|)$ dominator edges to center of stars.
4. There are $O(c^3|M|)$ dominator edges other than what we count.
5. Total number of dominators is in $O(c^3 \cdot t|M|)$.
6. Number of communication rounds is $O(1)$.
7. The original approximation factor is $O(c^2)$, for similicity we show this proof.

Class of graphs of genus at most $g$ has constant edge density and excludes $K_{3,4g+3}$.

□

# Plan

# Intuitive Idea

1. Find a canonical subgraph $K_v$ for each vertex $v$: Graphs where they are minimal on some circumstances and $K_{3,3}$ is a depth-one minor of $K_v$.
2. Remove all disjoint canonical graphs.
3. recurse for $g$ times.

# Intuitive Idea

1. If we remove $g$ times a $K_{3,3}$ minors the remaining graph is planar
2. If it is impossible to remove any depth-one minor $K_{3,3}$ at step $i \leq g$ then, the graph is locally embeddable.
3. There are at most $g$ disjoint $K_{3,3}$ minors in graph of genus $g$.
4. Run the normal algorithm on modified graph for $t := 3$.
5. Similar analysis shows that it is $24g + O(1)$-approximation.
6. Bad news: Number of communication rounds is $12g + O(1)$.

# Plan

1. Is it possible to improve the constant factor?
2. What is the biggest class of graphs which admits a constant factor approximation for MDS?

# Thank you