When is liquid democracy possible?

On the manipulation of variance

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Liquid democracy is a transitive vote delegation mechanism over voting graphs. It enables each voter to delegate their vote(s) to another better-informed voter, with the goal of collectively making a better decision. The question of whether liquid democracy outperforms direct voting has been previously studied in the context of local delegation mechanisms (where voters can only delegate to someone in their neighbourhood) and binary decision problems. It has previously been shown that it is impossible for local delegation mechanisms to outperform direct voting in *general graphs*. This raises the question: for which classes of graphs do local delegation mechanisms yield good results?

In this work, we analyse (1) properties of specific graphs and (2) properties of local delegation mechanisms on these graphs, determining where local delegation actually outperforms direct voting. We show that a critical graph property enabling liquid democracy is that the voting outcome of local delegation mechanisms preserves a sufficient amount of variance, thereby avoiding situations where delegation falls behind direct voting[1]. These insights allow us to prove our main results, namely that there exist local delegation mechanisms that perform no worse and in fact quantitatively better than direct voting in natural graph topologies like complete, random d-regular, and bounded degree graphs, lending a more nuanced perspective to previous impossibility results.

1 INTRODUCTION

Liquid democracy is a process in which voters are allowed to transitively delegate their votes to other voters in a flexible way. In other words, a voter (Alice) in a liquid democracy system has the choice to cast a vote directly to decide the issue at hand, or to delegate their vote to some other voter (Bob), who can in turn delegate their vote to someone else (Carol). In doing so, and assuming Carol does not delegate her vote to someone else, Carol's eventual vote effectively represents the votes of Alice, Bob, and Carol herself. Recently, liquid democracy has become an increasingly popular mechanism in governance structures and has been adopted by actual political parties like the German Pirate Party [25], corporate governance [21], blockchain decentralised autonomous organisations (DAOs) [14, 16], as well as centralised financial investment firms [29].

On the theoretical front, one of the driving questions behind liquid democracy is whether delegation actually increases the probability of making a better decision. A natural model in the literature [24] is to assume a binary setting where there is a correct voting outcome (say 1). Each voter v_i has a corresponding competency score $p_i \in [0, 1]$, which represents their probability of voting for the correct outcome. The social network of voters can be modeled as a graph, and commonly studied delegation mechanisms are *local delegation mechanisms*, which are mechanisms that only allow delegation to voters in the graph that are adjacent to any given voter.

[1]One can view our work as providing mathematical support for the disadvantages of a dictatorship: delegating all the votes to a single dictator will, in this model, lead to worse outcomes.

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In order to measure the performance of local delegation mechanism over direct voting, we can consider the difference in probability between delegation and direct voting with regards to deciding on the correct outcome (also known as "gain" in related work). In their seminal paper, Kahng et al. [24] answered the aforementioned question in the negative: over *general* graph topologies, no local delegation mechanism compared to direct voting can fulfill the following properties at the same time: (1) (*positive gain*) asymptotically achieve higher gain in certain graph topologies, and (2) (*do no harm*) negative gain asymptotically goes to zero over all topologies. Their negative result is based on a graph topology where voters delegate their votes (transitively or not) to a very small set of final delegates. This concentrates power in the hands of a few voters, and this higher degree in correlation of votes can lead to a violatation of the "do no harm" principle. We depict the star as an extreme example of this in Figure 1.

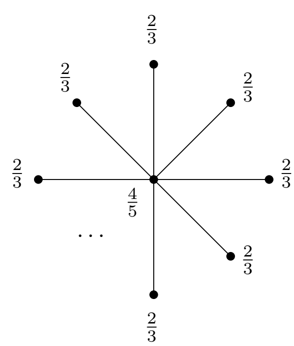

Fig. 1. Voters in a star topology labeled by competencies. The central node has a high competency of $\frac{4}{5}$ and each leaf node has a lower competency of $\frac{2}{3}$. We assume a delegation mechanism that delegates votes to strictly more competent voters. The probability of voting for the correct correct outcome in the direct voting setting converges to 1 asymptotically as the graph grows in size. However, delegation concentrates all voting power to the central node. Thus, the probability of voting correctly in the delegation setting is $\frac{4}{5}$ leading to a negative gain of $\frac{1}{5}$.

However, there are also first positive results. In particular, is has been shown that in a complete graph, if we assume some restrictions on the delegation mechanisms and the competency vector distribution, liquid democracy can achieve both probabilistic variants of positive gain and do no harm. In their influential work, Halpern et al. [20] assume voters can delegate to anyone (i.e., the voters form a complete graph topology), and the voters' competencies are sampled from some probability distribution. They go on to show that there are some classes of distributions and delegation mechanisms that satisfy probabilistic positive gain and do no harm. Interestingly, the main technical insight of the work of [20] corroborates the insight of [24] in the sense that they also show that a sufficient condition on the complete graph to preserve the do no harm principle is to ensure that the delegation mechanism does not delegate too much votes to any single voter.

Given the negative result for general graph classes and the positive result on the complete graph, we ask the fundamental question in this paper: *how does the graph topology affect the feasibility of liquid democracy?*

1.1 Our contribution

Specific graph topologies. Our work initiates the study of which graph properties enable efficient liquid democracy voting. Our main insight is that for many natural graphs liquid democracy is actually possible and outperforms direct voting. The graph topologies that we show positive results for are topologies like the complete graph, random d-regular graphs, and bounded degree and bounded minimal degree graphs, which are natural and well-studied [13, 34] topologies that are of both practical and theoretical interest. Additionally, the delegation mechanisms we analyse are simple and local in nature: they only look at the local neighbourhood of a node and delegate the vote to a random voter in the neighbourhood with higher competency. We stress that focusing on connectivity and local mechanisms allows the results of our work to apply to various realistic settings in which liquid democracy could be implemented where limited connectivity and locality is paramount (for instance, in corporate or social network settings where voters might be unwilling to delegate to users that are unfamiliar to them a priori).

Graph properties, strong positive gain, and recycle sampling. In contrast to prior work, we define a set of graph properties (e.g., completeness, bounded degree etc.) as well as an effect of a delegation mechanism over graphs that satisfy the graph properties. Although positive gain is typically an easier property to prove compared to do no harm, we define a novel and *stronger* notion of positive gain, namely strong positive gain, that holds *for all* instances of graphs and local delegation mechanisms that satisfy a subset of the above properties and delegation effects. Additionally, we quantify the exact amount of increase in expectation achieved by our mechanisms over direct voting. The key technical tool involved in proving strong positive gain is a novel model of dependent random variables called *recycle sampling* (see Section 3.1) that captures the dependency structure in our delegation mechanisms (i.e., that the outcome of a voter is correlated with the outcomes of voters' with higher competencies). We then leverage this specific dependency structure to lower bound the sum of these dependent random variables from their expectation (Lemma 5).

Variance and do no harm. As observed in the star example in Figure 1, the do no harm property can be violated when the voting power concentrates around a small subset of voters. This phenomenon is also observed in prior work, which circumvent the issue by either using non-local mechanisms [24] or ensuring that the maximum weight of any single voter is bounded [20]. Our work presents two complementary but distinct sufficient conditions that preserves the do no harm property for the classes of graphs we study (see Section 3.2). The commonality between these conditions is that they ensure sufficient amount of variance in the voting outcome such as to avoid concentrating power in the hands of a few voters. The first sufficient condition is that the competencies of all voters lie in an interval strictly bounded away from the extremal values of 0 and 1 and that the total amount of delegation for any local mechanism is less than $n^{\frac{1}{2}-\varepsilon}$ for all $\varepsilon > 0$ (Lemma 2). We stress that our result holds for local mechanisms, closing the gap introduced by the result of [24]. The second condition is an upper bound on the maximum number of votes delegated to any voter in *general graphs* (Lemma 3). Although this result is similar to prior work [20], we note that their result was derived in the setting of complete graphs, whereas our result holds for general graph topologies.

Implications. Altogether, our results imply that the best classes of graphs for liquid democracy are graphs that do not have too much structural asymmetry in the node degrees. Towards a widespread and realistic adoption of liquid democracy, it would be interesting for future work to empirically examine our graph properties and the variance inducing conditions specified in Lemmas 2 and 3 in various real-world networks to see if our conditions are justified in real-world settings.

1.2 Related work

Delegated and proxy voting is a well-studied area of research [1, 28, 32]. Recently, several works [5, 6, 9, 20, 24] have focused on the question of when is delegated voting better than direct voting, showing that delegated voting increases the mean voting outcome. The work of [24], however, was the first to propose an algorithmic model and analysis of local delegation mechanisms in general graphs. In particular, [24] shows that although in general graphs there will always be examples where delegated voting leads to a better outcome compared to direct voting, there also exist cases where delegation leads to worse outcomes, particularly when voters concentrate their delegations on a few candidates. Our work is motivated by the model and negative result of [24], whereby we seek to answer a separate but complementary question of which types of graphs guarantee better performance of delegation mechanisms and the reasons why this is the case. Our work is also closely related to the work of [20] which focuses on the setting where voters' competencies are not fixed but sampled from a distribution and they study distributions and delegation mechanisms that guarantee to perform better than direct voting. In contrast, our work focuses on graph topologies rather than competency distributions (i.e., we focus on connectivity assumptions between voters rather than voters' competencies), and thus is a complementary but orthogonal line of research.

Empirical analyses of liquid democracy have been conducted in actual political [7, 25, 33] and blockchain [3, 14, 16, 19, 23, 27] settings. Most closely related to our work are recent empirical studies that highlight a concentration of voting power [25, 31], as well as the work of [17] that aims to minimise the amount of votes sent to the maximum weighted voter in a liquid democracy voting system. Our work solidifies the assumptions in these works that too much weight to a single voter is adverse to the system, and is the first to formally show this for general graphs.

Further upfield but also related are works that look at liquid democracy from a rational perspective, examining the question of when would delegation maximise a voter's utility [8, 18]. [4, 35] adopt a game theoretic perspective, showing the existence of delegation strategies that form Nash Equilibria. These are complementary directions that look at the decision making from a single agent perspective, whereas our work focuses on cooperative decision making.

The interaction model considered in this paper is restricted to a voter's neighbourhood, in the spirit of the LOCAL model in distributed computing [26, 30], in which among many other problems, other types of voting problems have also been studied, such as the spread of influence in social influence networks [15]. The main difference between our local mechanisms and the LOCAL model is that our mechanisms do not require voters to know the total number of voters, but instead need to know the set of voters in their local neighbourhood that are more competent than themselves.

2 PRELIMINARIES

2.1 Model

Problem. We consider the problem of vote delegation initially proposed by [24]. In this model, we have n voters that want to decide on some binary issue by voting. We assume the existence of a ground truth with regards to the binary question that is unknown to the voters. Each voter v_i has some competency level $p_i \in [0, 1]$ that denotes the probability that v_i votes correctly on the issue.

We define an instance of our problem as a graph $G = (V, E, \mathbf{p})$. V represents the set of voters, and an edge between any two voters v_i and v_j represents the fact that v_i and v_j are aware of each other. The competency p_i of each voter v_i is denoted by the ith element of the vector of competencies $\mathbf{p} = [p_1, p_2, \ldots, p_n]$. Wlog, we order the voters by their probability such that $p_i \leq p_j$ if $i < j$.

Graph restrictions. In our work, we focus on certain classes of graphs that give rise to positive results for delegation mechanisms over direct voting. These are graphs that are restricted to satisfy certain predefined *graph properties.* Formally, we define a restriction of the problem setting on n vertices as a graph restriction. Let $\mathcal{G}_n$ denote the set of all graphs on n vertices (with corresponding competency vector).

Definition 1. *(Graph restriction.) A* graph restriction $\mathcal{G}_n^{\mathcal{P}} \subset \mathcal{G}_n$ *for a set of graph properties* $\mathcal{P}$ *is the set of graphs* $G = (V, E, \mathbf{p})$ *such that for all* $G \in \mathcal{G}_n^{\mathcal{P}}$, G *satisfies all properties in* $\mathcal{P}$.

For $(V, E, \mathbf{p})$, *we use the following graph restrictions:*

- K_n, *the graph* (V, E) *is a complete graph;*

- $\text{Rand}(n, d)$, *the graph* (V, E) *is a random graph with degree* d, *generated after* $\mathbf{p}$ *is assigned;*
- $\Delta \leq k$, *the largest degree of graph* (V, E) *is at most* k*;*
- $\delta \geq k$, *the smallest degree of graph* (V, E) *is at least* k*;*
- $\text{PC} = a$, *the sum of competencies satisfies* $\frac{1}{2} > \frac{1}{n}\Sigma_{i=1}^{n} p_i > \frac{1}{2} - a$*;*
- $\mathbf{p} \in (\beta, 1 - \beta)$, *all competencies lie in range* $(\beta, 1 - \beta)$ *for some* $\beta \in (0, \frac{1}{2})$*.*

The first four restrictions are pure graph-theoretical restrictions. We call $\text{PC} = a$ the plausible changeability. It intuitively captures the idea that the competency vector in the problem instance is "sufficiently close" on average to $\frac{1}{2}$. Hence, given enough instances of delegation, a delegation mechanism can change the probability of the voting outcome. We call restriction $\mathbf{p} \in (\beta, 1 - \beta)$ bounded competency. It requires that no voter is either completely incompetent or competent at deciding on the underlying issue. We note that this restriction is also present in [24].

Available information. Given a parameter $\alpha > 0$, for every voter i, we define the set of approved voters $J(i)$ to be the set of voters with indices j, such that $p_i + \alpha \leq p_j$. We assume that each voter only knows the (pseudonymous) identity of their neighbouring nodes. Moreover, each voter also knows which neighbours are approved. We stress that we do not assume voters know competencies of their own or anyone else, and all the voters from the set of approved voters are indistinguishable.

2.2 Mechanisms

A delegation mechanism is a function that takes a problem instance G as input and outputs for each voter a probability distribution over who to delegate their vote to (or none, in which case the voter simply votes directly). We focus on *local delegation mechanisms*, which make the decision only based on the approval set and an arbitrary ranking over the voters in this set. We describe two examples of local delegation mechanisms below.

Example 1. *(Delegation based on approval set size.) Let* M *be a mechanism that checks if* $|J(i)| > j$ *for a threshold* j *and then delegates the vote of* v_i *to a random voter in* $J(i)$ *whenever the check returns true. Then* M *is a local delegation mechanism.*

Example 2. *(Direct voting.) Let* D *be the mechanism that does not delegate votes for all voters. That is, each voter simply votes directly for themselves. Then* M *is also a local delegation mechanism.*

Delegation. To evaluate the probability of making a correct decision for a delegation mechanism M, we first apply the mechanism M to the problem instance G. For each voter, we sample delegates from the probability distribution output from M to get a directed *delegation graph* where a directed edge between voters (v_i, v_j) denotes that v_i delegates their vote to v_j. Observe that all the delegation mechanisms that delegate the vote to the approved set create *acyclic* delegation graph since $\alpha > 0$. We say a delegation mechanism M is *acyclic* if all possible induced delegation graphs from M are acyclic (up to self cycles). An example of a possible delegation graph corresponding to a delegation mechanism from Example 1 on a problem instance is given in Figure 2.

Probability of correct decision. To evaluate the probability of making a correct decision for a delegation mechanism M, we first apply the mechanism M to the problem instance G. Let S denote the set of sinks from the delegation graph output of M. We assign each sink v_i a corresponding weight w_i denoting the number of votes delegated to v_i (including self-votes). For each $v_i \in S$, v_i votes for the correct option with probability p_i. Finally, a decision is made based on weighted majority vote, i.e., let $S' \subset S$ denote the voters that voted for the correct option. Then, the correct option will be chosen only if $\sum_{v_i \in S'} w_i \geq \sum_{v_i \in S \setminus S'} w_i$. We denote the *probability* that mechanism M outputs the correct decision for a problem instance G by $P^M(G)$.

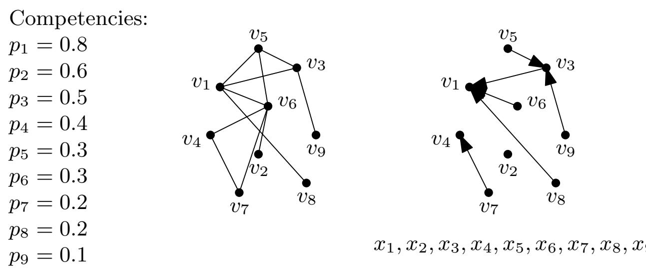

Fig. 2. The graph on the left depicts the problem instance over a set of 9 voters $v_1, \ldots, v_9$ and their corresponding competencies. We assume the parameter $\alpha = 0.01$. The graph on the right depicts a possible delegation graph corresponding to the output of a local delegation mechanism (as in Example 1 with threshold $j = 0$) on the problem instance. The x_is below the graph on the right depicts the random variables representing the voting outcomes of each voter.

Mechanism gain. Let M denote a local delegation mechanism on some problem instance G and let D denote direct voting over the same instance. The *gain* of M over D on instance G is defined as $\text{gain}(M, G) = P^M(G) - P^D(G)$. We use *loss* to denote negative gain.

Delegate restriction. Finally, in some results, we also restrict the evaluation of our delegation mechanisms to mechanisms that, when acting over a set of input graphs, satisfy the property that at least some number of voters delegate their vote. Looking ahead, this delegate restriction property is necessary to ensure that enough votes are delegated to more competent voters to increase the gain of the mechanism over direct voting.

Definition 2. *(Delegate restriction.) Let $\mathcal{M}$ be a set of mechanisms, and $\mathcal{G}^{\mathcal{P}}$ be a set of graphs. Then $\mathcal{M}, \mathcal{G}^{\mathcal{P}}$ satisfy* $\text{Delegate}(n) > f(n)$ *if for all $G \in \mathcal{G}^{\mathcal{P}}$ and all $M \in \mathcal{M}$ at least $f(n)$ voters delegate.*

2.3 Desiderata

Here we define two important desiderata that our mechanisms need to satisfy.

Do no harm (DNH). The first desideratum we want our mechanisms to satisfy is do no harm. Intuitively, this ensures that as our restricted problem instances grow in size, the loss of our mechanisms compared to direct voting goes to 0.

Definition 3. *(Do no harm.) A mechanism M and set of graph properties $\mathcal{P}$ satisfies the do no harm property if for every $\varepsilon > 0$, $\exists n' \in \mathbb{N}$ such that for all graphs $G \in \mathcal{G}_n^{\mathcal{P}}$ where $n > n'$, we have $gain(M, G) > -\varepsilon$.*

Strong positive gain (SPG). The second desideratum that we want our mechanisms to achieve is strong positive gain. In [24], positive gain (defined in Appendix A.1) ensures that as problem instances grow in size there are some instances where delegated mechanisms perform better (i.e., achieve positive gain) compared to direct voting. We define a more restrictive version of positive gain, *strong positive gain (SPG)*, to hold not just when our mechanisms on *some* large enough problem instances perform better compared to direct voting, but for *all* problem instances. Typically, we prove SPG for all instances that in addition satisfy $\text{Delegate}(n) > f(n)$ for some function of the total number of voters $f(n)$.

Definition 4. *(Strong positive gain.) A mechanism M, a set of graph properties $\mathcal{P}$, and a function $f(n)$ satisfies the strong positive gain property if there exists $\gamma > 0$ and $n' \in \mathbb{N}$ such that for all $n > n'$, for all $G \in \mathcal{G}_n^{\mathcal{P}}$ such that (M, G) satisfies* $\text{Delegate}(n) \geq f(n)$*, we have $gain(M, G) > \gamma$.*

Note that strong positive gain is a stronger notion of positive gain as designing mechanisms that satisfy positive gain as in Definition 6 only shows the existence of problem instances of a certain size where delegation mechanisms perform better compared to direct voting. In contrast, designing mechanisms that satisfy strong positive gain constructs an entire class of problem instances of a certain size that satisfies positive gain.

3 PRINCIPLES OF LIQUID DEMOCRACY: SAMPLING AND GRAPH PROPERTIES

In this section, we present the core lemmas that we use in Sections 4 and 5 to show SPG and DNH for certain classes of graphs. In Section 3.1, we define a novel technique to bound the sum of dependent random variables to show SPG for complete and random d-regular graphs. Unlike the classic extension of the Chernoff bound to negatively correlated random variables [12], the random variables in our setting are positively correlated and capture the specific dependency of the outcomes of votes in the delegated setting. In Section 3.2, we define two core graph properties that ensure sufficient variance in the voting outcome and thus serve as sufficient conditions for DNH.

3.1 A new notion of sampling and bound on the sum of dependent random variables

A crucial feature of liquid democracy and delegation is that delegating votes fixes one's voting outcome to that of the voter one delegates to. This then creates a certain *dependency* between the outcomes of the votes. Here we present a way to model the specific dependency of the outcomes of votes in the delegated setting and bound the corresponding sum of correct votes. We are not aware of an existing model of such dependencies and a method to bound the sum of variables that depend on each other in this particular way, so we believe that this is a result of independent interest.

We define a new sampling notion in Definition 5 called *recycle sampling* that captures the dependency in the delegated voting process, as well as a way to measure the degree of dependency among the random variables. We then prove in Lemma 1 that the sum of the correct votes is close to its expectation as long as the degree of dependency is not too high. Looking ahead, Lemma 1 is used in Section 4 to show SPG for local delegation mechanisms restricted to the complete graph and random d-regular graphs problem settings.

To model the dependency of the delegated voting process, we first define the recycle sampling graph that captures the delegation outcome in an abstract manner. The recycle sampling graph is defined for Bernoulli random variables where the Bernoulli parameter can either sampled from a set of previously known parameters corresponding to some random variables (hence the parameter is "recycled"), or a fresh parameter is chosen. This captures probability of voting correctly in the delegation setting where this probability (the Bernoulli parameter) depends on whether one votes directly or delegates to someone else (and hence recycles their Bernoulli parameter).

Definition 5. *We say G is a (j, c, n)-recycle sampling graph if*

- *vertices are ordered $\{v_1, v_2, \ldots, v_n\}$ and for every $i > j'$ for some $j' \geq j$, we have a directed edge (v_i, v_k) for all $k \in [i']$ where $i' \geq j$,*
- *vertices are labelled by (z_i, p_i) and represent random variables $x_1, \ldots, x_n$ where x_i with probability $1 - z_i$ takes the value of a random successor and with probability z_i, it is Bernoulli random variable with parameter p_i.*
- *the longest path in G (sometimes called partition complexity) has length at most c.*

Let $X_n = \sum_{i=1}^{n} x_i$ be the recycle sampling random variable associated with G. It is the outcome of realizing G: for increasing i, x_i is either (with probability z_i) the outcome of a random variable with parameter p_i; or it is equal to a randomly selected successor.

Intuitively, the parameter c captures the *degree of dependency* introduced among the random variables from the process. A larger c indicates a larger degree of dependency. We note that c depends on the approval threshold α and a simple upper bound for any mechanism is $\frac{1}{\alpha} \leq c$.

Lemma 1. *Let X_n be a random variable associated with G, (j, c, n)-recycle-sampled graph, then we have with probability at least $1 - e^{-\Omega(j^{1/3})}$*

$$X_n \geq \mu(X_n) - c\frac{\varepsilon \cdot n}{j^{1/3}}.$$

Proof idea and outline. The goal in the proof is twofold: (1) to create a modified sequence of independent random variables $\hat{X}_n = \{\hat{x}_1, \ldots \hat{x}_n\}$ such that we can use Lemma 5 (Chernoff bound) to bound the probability that $\Sigma_{i=1}^n \hat{x}_i$ deviates too far below $\mu(X_n)$. We do so partition by partition (sets where there is no edge between two vertices): we modify the Bernoulli parameter of some of the original random variables x_i in a single partition such that the parameter does not depend on the variables in predeseccor partitions. This eventually decorrelates all variables in the modified sequence. (2) The way we modify the parameter takes into account the worst-case behaviour (i.e., more incorrect votes than expected) that could happen in the predeseccor partition. This means that each time we make the modification we decrease the concentration lower bound of the modified sequence (i.e., make the lower tail of the distribution of $\hat{X}_n = \sum_i^n \hat{x}_i$ fatter). However, for a constant number of partitions c, this will only decrease the concentration lower bound by a factor of $c\frac{\varepsilon \cdot n}{j^{1/3}}$. The full proof of Lemma 1 is in Appendix B.2.

3.2 Transmutation[2]: manipulating variance to ensure DNH

Here we present two core graph properties that, together with a reasonable delegate restriction, provide sufficient conditions that ensure delegation mechanisms satisfy DNH. Informally, a common thread running through both properties is that they ensure sufficient variance in the voting outcome. This enables the delegation mechanisms to avoid the voting outcomes where the overall decision hinges on the choices of a few influencial voters, which could lead to settings where delegation performs worse than direct voting (e.g., in the example in Figure 1).

3.2.1 Anti-concentration for $\mathbf{p} \in (\beta, 1-\beta)$. We show a connection between the assumption that all voters' competencies are bounded away from the extremal values of 0 and 1 and the DNH property. We prove that any delegation mechanism (both local and non-local) over any problem instance satisfies the DNH as long as the number of delegated votes is sufficiently small. The key insight is that $\mathbf{p} \in (\beta, 1-\beta)$ guarantees a sufficient amount of variance in the outcome of the direct voting setting, such that if only small number of voters delegate the loss of the mechanism (i.e., the harm) goes asymptotically to 0. We formally state our result as follows.

Lemma 2. *For all $\varepsilon > 0$ and all $\mathcal{P}$ where $(\mathbf{p} \in (\beta, 1-\beta)) \in \mathcal{P}$, any mechanism that delegates less than $n^{\frac{1}{2}-\varepsilon}$ votes satisfies DNH.*

Proof idea and outline. We use the fact that the distribution of the total number of correct votes in the direct voting setting with bounded competencies (denoted by X_n^D) converges asymptotically to the Normal distribution with known mean and variance. This allows us to show anti-concentration bounds on the expected value of X_n^D. The full proof of Lemma 2 is in Appendix B.3.

[2]Definition from the DnD 5e handbook: "You are a student of spells that modify energy and matter".

3.2.2 Maximum weight of a single voter in general topologies. The second condition is a bound on the maximum weight of any voter as a result of delegation. Intuitively, a small maximum weight guarantees that there will be enough sinks in the delegation graph. This consequently generates sufficient variance in the voting outcome to preserve DNH.

Lemma 3. *Let X_n be the voting outcome any delegation mechanism where every sink has weight at most w. Then with probability $1 - e^{-\Omega(n^{\varepsilon})}$ for any constant $c > 0$ we have*

$$|\mu(X_n) - X_n| \leq \frac{1}{c}\sqrt{n^{1+\varepsilon} w}\,.$$

Note that Lemma 3 implies that the maximum weight $w < n^{1+\varepsilon}$. The proof of the lemma is a simple consequence of using the largest weight to lower bound the number of sinks, and then use Hoeffding's inequality. The full proof of Lemma 3 is in Appendix B.4.

4 BREAKING THE IMPOSSIBILITY FOR CERTAIN GRAPHS

We now proceed to analyse two classes of graphs, complete and random d-regular graphs, and propose local delegation mechanisms that achieve SPG and DNH on both these classes of graphs.

4.1 Complete graphs

In this section, we propose a delegation mechanism (detailed in Algorithm 1) that describes the delegation mechanism for a single voter. For a voter v_i, we check the size of the approval set $|J(i)|$ to compare it with a threshold $j(n)$. The argument of j is the number of neighbours of v_i which, in the complete graph, is equivalent to the total number of voters n. If $|J(i)| > j(n)$, v_i delegates their vote to a randomly selected approved neighbour from $J(i)$. Otherwise, v_i casts a vote which will be correct with probability p_i. The guarantees provided by Algorithm 1 are good for small $j(n)$ (even $j(n) \in o(n)$), since we want the maximum number of votes to be delegated (such that $\mathcal{M}$ satisfies Delegate$(n) > \frac{n}{k}\}$). Looking ahead, the proof of DNH in Lemma 4 assumes $j(n) \leq \frac{n}{3}$.

Algorithm 1 Delegation algorithm for a single voter v_i in K_n

Input: number of neighbors n, a set of approved voters $J(i)$, threshold function j
Output: decisions to vote or delegate to another voter

```
if J(i) ≥ j(n) then
    v_f ← RandomChoice(J(i))
    Delegate(v_f)
else
    Vote(v_i)
```

Let V_i denote the set $\{v_1, \dots, v_i\}$, that is, the top i voters according to competency level. Let $Y_n = \{y_i, \dots, y_n\}$ denote the corresponding sequence of random variables representing the outcomes of the votes *as output by Algorithm 1* on each of the voters in the sequence V_n. That is, y_i denotes the random variable representing the outcome of voter v_i. We let $Y := \Sigma_{i=1}^{n} Y_i$ denote the sum of correct votes in the sequence Y_n. Recall that we use X_n to denote the sum of an independent sequence of Bernoulli random variables $\{x_1, \dots, x_n\}$ with Bernoulli parameters $\{p_1, \dots, p_n\}$. We can think of the sequence $\{x_1, \dots, x_n\}$ representing the outcomes of the voters $v_1, \dots, v_n$ in the direct voting setting (i.e., unmodified by Algorithm 1).

Lemma 4. *(Increase in expectation.) Let Y_n be sequence given by Algorithm 1 where k voters do not delegate, we have*

$$P[Y > \mu(X_n) + (n-k)\alpha - \frac{1}{\alpha}\frac{\varepsilon n}{j(n)^{1/3}}] > 1 - e^{-\Omega(j(n)^{1/3})} \ .$$

Proof idea and outline. The proof of Lemma 4 makes use of two key insights. First, Y_n forms a sequence of $(j(n), \frac{1}{\alpha}, n)$-recycle-sampled random variables. Thus, we can use Lemma 1 to bound the sum of the variables Y against the expected value of Y. The second insight is that every act of delegation as specified by Algorithm 1 increases the expected value of Y by at least α, which gives us a way to compute the expected value of Y given a bound of the number of voters that do not delegate their votes. The full proof of Lemma 4 is given in Appendix B.5.

As a direct consequence of Lemma 4, we get the following, which gives us SPG for the mechanism described in Algorithm 1 with graph properties $\mathcal{P}$ and mechanism-graph property $\mathcal{P}_2$.

Theorem 1. *Algorithm 1 satisfies*

- *SPG for graph properties* $\mathcal{P} = \{K_n, \mathrm{PC} = \frac{\alpha}{k}\}$ *and* $\mathrm{Delegate}(n) > \frac{n}{k+\varepsilon}$ *for some* $k > 1$ *and* $\varepsilon > 0$,
- *and DNH for* $\mathcal{P} = \{K_n\}$.

Proof idea and outline. The first result is a consequence of Lemma 4. It shows the increase in expectation of the delegation mechanism when there are at least $\Omega\left(\frac{n}{j(n)^{1/3}}\right)$ delegations. If at least $\frac{n}{k+\varepsilon}$ voters delegate, in the case satisfying $\mathrm{PC} = \frac{\alpha}{k}$, then the outcome is 1 with high probability. For DNH, the cases when a lot of voters delegate or when the competencies are bounded away from 0 and 1 are handled by Lemmas 2 and 4. For those remaining, we show that the outcome of direct voting and delegation aligns with high probability. The proof of Theorem 1 is in Appendix B.6.

4.2 Random d-regular graphs

Algorithm 2 describes the creation of $\mathrm{Rand}(n, d)$ as well as the delegation mechanism. In the algorithm, every voter v_i first samples d random neighbours. Then v_i checks if at least $j(d)$ of these d neighbours are in its approval set. If so, v_i delegates their vote to a randomly approved neighbour.

Algorithm 2 Algorithm for a single voter v_i.

Input: parameter d, function j, sampling function RandomNeighbours
Output: decisions to vote or delegate to voter

```
D ← RandomNeighbours(d)                          ▹ Selects d random voters.
J(i) ∩ D ← Approved(D, p_i)                      ▹ Selects approved voters
if J(i) ∩ D ≥ j(d) then                          ▹ j(d) is a fraction of d.
    v_f ← RandomChoice(J(i) ∩ D)
    Delegate(v_f)
else
    Vote(v_i)
```

Theorem 2. *Algorithm 2 satisfies*

- *SPG for graph properties* $\mathcal{P} = \{\mathrm{Rand}(n, d), \mathrm{PC} = \frac{\alpha}{k}\}$ *and* $\mathrm{Delegate}(n) > \frac{n}{k+\varepsilon}$ *for some* $k > 1$ *and* $\varepsilon > 0$,
- *and DNH for graph property* $\mathcal{P} = \{\mathrm{Rand}(n, d)\}$.

Proof idea and outline. The key ingredient to the proof is realising that the situation on the random graph is similar to the complete graph with threshold $\frac{j(d)}{d}n$. Then, we have all delegation happening not surely, but in expectation. This, with a small modification, allows us again to use recycle sampling. The full proof of Theorem 2 is in Appendix B.7.

5 GENERAL GRAPHS: GRAPHS WITH BOUNDED DEGREE

In this section, we extend our analysis to general graphs with bounded degree and minimal degree. We show local delegation mechanisms on these classes of graphs that achieve SPG and DNH.

5.1 Bounded degree

THEOREM 3. *Let $\mathcal{M}$ be any delegation mechanism, then $\mathcal{M}$ satisfies*

- *SPG for properties $\mathcal{P} = \{\Delta \leq t^{\alpha/(1+\varepsilon)}, \mathrm{PC} = \frac{t\alpha}{2n}\}$ and $\mathrm{Delegate}(n) > t$ for any t and $\varepsilon > 0$,*
- *and DNH for properties $\mathcal{P} = \{\Delta \leq n^{\alpha/(2+\varepsilon)}, \mathbf{p} \in (\beta, 1-\beta)\}$.*

Proof idea and outline. The main idea for SPG is to use the maximum degree to bound the length of the longest delegation path. This and the maximum degree bounds the total weight of any sink, which allows us to use Lemma 7 to guarantee a sufficient increase in expected value in order to show SPG. The proof of DNH follows as a consequence of both Lemma 2 and a positive gain, depending on the number of delegations. The full proof of Theorem 3 is in Appendix B.8.

5.2 Bounded Minimal Degree

THEOREM 4. *Let $\mathcal{M}$ be a delegation mechanism where a voter delegates if at least $\frac{2}{3}$ of its neighbors are approved. Then, $\mathcal{M}$ satisfies*

- *SPG for properties $\mathcal{P} = \{\delta \geq n^{\varepsilon}, \mathrm{PC} = \frac{\alpha h}{2n}\}$ and $\mathrm{Delegate}(n) > h$ for any $h \geq \sqrt{n}$ and $\varepsilon > 0$,*
- *and DNH for properties $\mathcal{P} = \{\delta \geq n^{\varepsilon}, \mathbf{p} \in (\beta, 1-\beta)\}$.*

Proof idea and outline. We first upper bound the maximum weight of the top $\frac{2\delta}{3}$ voters in terms of competency as well as use the Chernoff bound to probabilistically bound the maximum weight of the most competent voter. We then show that our choice of parameters results in the maximum weight being small enough to use Lemma 3 to show that the total number of correct votes in the delegated setting is close to its expectation. The full proof of Theorem 4 is in Appendix B.9.

6 DISCUSSION, EXTENSIONS, AND IMPLICATIONS

General graphs structures and structural symmetry. Our results from Sections 4 and 5 show that the types of graphs that yield the best results for delegation over direct voting are graphs that do not have too much structural asymmetry in terms of degrees among nodes. A direct and obvious implication of our results is that concentration of influence could lead to harmful decision making. An interesting and perhaps less obvious follow up question is the connection between uniform weights and network connectivity in ensuring that harmful outcomes do not happen with delegation. We believe it would be interesting to think about and further explore the social implications of this (perhaps in relation to spreading misinformation in social networks [11]). We leave this as a direction of future work.

Weighted majority vote. The delegation mechanisms proposed so far in our work are extremely simple and rely only on the knowledge of the delegates in a voter's approval set and a random choice of a voter in this approval set to delegate to. We conjecture that our model and analysis can be extended to the more sophisticated weighted majority vote setting, as they are emerging on certain blockchains [10]. In this setting, a voter delegates their vote to some number of approved delegates, and then takes a weighted majority of the delegated votes using some locally defined weight function over the delegates[3]. It is easy to see that as long as voters only delegate to more competent delegates, our analysis and results on strong positive gain will also transfer to the weighted majority setting (it is similar to sampling the random delegate multiple times and taking the best outcomes in the setting where voters can only delegate to one random other voter in their approval set). For DNH, we conjecture that as long as the weighting function together with the mechanism satisfies Lemma 3, we would also be able to ensure sufficient variance in the voting outcome to preserve DNH.

Vote abstaining. Our model and analysis can be easily extended to the setting where voters can abstain from voting or delegating their votes. Specifically, we examine the abstinence model where a voter can abstain from voting *only if* they can delegate their vote to someone else. This models *decision-agnostic* behaviour where voters that do not particularly care about the issue at hand can simply not vote instead of entrusting their vote to someone else, corroborating the empirical analysis of [19] that these decision-agnostic voters form a large majority of voters in liquid democracy voting systems. We first observe that this model of abstaining preserves the DNH property. This is because DNH typically gets violated when there is too much delegation (which runs the risk of too many votes being delegated to a few voters)[4]. We note that our analysis and results of strong positive gain also extends to this model of abstaining, again under a similar delegate restriction that some amount of delegation has to occur. We emphasise though, that the amount of gain guaranteed would likely be smaller compared to the setting where abstaining is not allowed.

Practical considerations in implementing liquid democracy. Lemmas 2 and 3 gives us sufficient conditions to guarantee delegation mechanisms on certain classes of graphs satisfy DNH. In the context of implementing liquid democracy in real world settings, an interesting implication would be to check if the assumptions specified in Lemmas 2 and 3 actually exist in reality. For instance, it would be interesting to empirically verify if social networks or even random graphs that model social networks (e.g., Barabasi Albert graphs [2]) satisfy the assumptions on the amount of sinks with not too much weight in Lemma 3. Additionally, in practice the vector of competencies will not be deterministic as in our model, but probabilistic (similar to the model in [20]). Extending our model and analysis to account for probabilistic competencies in addition to classes of graphs would be an interesting and important step towards implementing liquid democracy delegation mechanisms in practice. Doing so would also unify our analysis on graph properties with the competency distributions analysis of [20] to present a coherent set of properties of both competency distributions and graph topologies in order to guarantee better delegation performance compared to direct voting.

7 CONCLUSION

In this work, we examined the question of whether there are some classes of graphs and delegation mechanisms in which liquid democracy outperforms direct voting. We show that for complete graphs, random d-regular graphs, bounded degree and bounded minimal degree graphs, as long as the total amount of delegation meets some minimal amount, the delegation mechanisms enjoy strong positive gain over direct voting. For the same classes of graphs, we also show that any local

[3]We note that any non-trivial weight function already assumes additional information about the delegates compared to our model assumptions as described in Section 2.1.

[4]In general, one has to be careful of how abstaining is defined. Allowing *all* voters the possibility of abstaining from voting could result in all but one sink abstaining and thus could violate DNH. However, if we limit the possibility of abstaining only to the voters that can delegate, as described in our model of abstaining, then we avoid such violations.

delegation mechanism will not cause harm. We believe our work opens up a wide range of future directions (discussed in Section 6) and we hope that our paper would pave the way for future work to tackle these issues.

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REFERENCES

[1] Dan Alger. 2006. Voting by proxy. *Public Choice* 126 (02 2006), 1–26. https://doi.org/10.1007/s11127-006-3059-1

[2] Albert-Laszlo Barabasi and Reka Albert. 1999. Albert, R.: Emergence of Scaling in Random Networks. Science 286, 509-512. *Science (New York, N.Y.)* 286 (11 1999), 509–12. https://doi.org/10.1126/science.286.5439.509

[3] Tom Barbereau, Reilly Smethurst, Orestis Papageorgiou, Johannes Sedlmeir, and Gilbert Fridgen. 2023. Decentralised Finance's timocratic governance: The distribution and exercise of tokenised voting rights. *Technology in Society* 73 (2023), 102251. https://doi.org/10.1016/j.techsoc.2023.102251

[4] Daan Bloembergen, Davide Grossi, and Martin Lackner. 2019. On Rational Delegations in Liquid Democracy. In *AAAI*. AAAI Press, 1796–1803.

[5] Gregory Butterworth and Richard Booth. 2023. A Contribution to the Defense of Liquid Democracy. In *Proceedings of the 24th Annual International Conference on Digital Government Research* (Gda?sk, Poland) *(dg.o '23)*. Association for Computing Machinery, New York, NY, USA, 244–250. https://doi.org/10.1145/3598469.3598496

[6] Ioannis Caragiannis and Evi Micha. 2019. A contribution to the critique of liquid democracy. In *Proceedings of the 28th International Joint Conference on Artificial Intelligence* (Macao, China) *(IJCAI'19)*. AAAI Press, 116–122.

[7] Alessandra Casella, Joseph Campbell, Lucas de Lara, Victoria Mooers, and Dilip Ravindran. 2023. *Liquid Democracy. Two Experiments on Delegation in Voting*. Research Papers. Columbia University. https://dx.doi.org/10.2139/ssrn.4307183

[8] Zoé Christoff and Davide Grossi. 2017. Binary Voting with Delegable Proxy: An Analysis of Liquid Democracy. In *Theoretical Aspects of Rationality and Knowledge*. https://api.semanticscholar.org/CorpusID:7451305

[9] Gal Cohensius, Shie Mannor, Reshef Meir, Eli Meirom, and Ariel Orda. 2017. Proxy Voting for Better Outcomes. In *Proceedings of the 16th Conference on Autonomous Agents and MultiAgent Systems* (São Paulo, Brazil) *(AAMAS '17)*. International Foundation for Autonomous Agents and Multiagent Systems, Richland, SC, 858–866.

[10] The Internet Computer. 2024. Following Rules. https://internetcomputer.org/docs/current/developer-docs/daos/nns/concepts/neurons/neuron-following#:~:text=Following%20rules%E2%80%8B,neurons%20for%20each%20proposal%20topic. Last accessed 7 February 2025.

[11] Sebastiano Della Lena. 2024. The spread of misinformation in networks with individual and social learning. *European Economic Review* 168 (2024), 104804. https://doi.org/10.1016/j.euroecorev.2024.104804

[12] Devdatt P. Dubhashi and Desh Ranjan. 1998. Balls and bins: A study in negative dependence. *Random Struct. Algorithms* 13, 2 (1998), 99–124.

[13] Guy Even, Moti Medina, and Dana Ron. 2015. Distributed Maximum Matching in Bounded Degree Graphs. In *Proceedings of the 16th International Conference on Distributed Computing and Networking* (Goa, India) *(ICDCN '15)*. Association for Computing Machinery, New York, NY, USA, Article 18, 10 pages. https://doi.org/10.1145/2684464.2684469

[14] Rainer Feichtinger, Robin Fritsch, Yann Vonlanthen, and Roger Wattenhofer. 2023. The Hidden Shortcomings of (D)AOs - An Empirical Study of On-Chain Governance. *CoRR* abs/2302.12125 (2023).

[15] Silvio Frischknecht, Barbara Keller, and Roger Wattenhofer. 2013. Convergence in (Social) Influence Networks. In *DISC (Lecture Notes in Computer Science, Vol. 8205)*. Springer, 433–446.

[16] Robin Fritsch, Marino Müller, and Roger Wattenhofer. 2024. Analyzing voting power in decentralized governance: Who controls DAOs? *Blockchain: Research and Applications* 5, 3 (2024), 100208. https://doi.org/10.1016/j.bcra.2024.100208

[17] Paul GÖlz, Anson Kahng, Simon Mackenzie, and Ariel D. Procaccia. 2021. The Fluid Mechanics of Liquid Democracy. *ACM Trans. Econ. Comput.* 9, 4, Article 23 (Oct. 2021), 39 pages. https://doi.org/10.1145/3485012

[18] James Green-Armytage. 2015. Direct voting and proxy voting. *Constitutional Political Economy* 26 (2015), 190–220. https://api.semanticscholar.org/CorpusID:154590189

[19] Andrew B. Hall and Sho Miyazaki. 2024. *What Happens When Anyone Can Be Your Representative? Studying the Use of Liquid Democracy for High-Stakes Decisions in Online Platforms*. Research Papers 4220. Stanford University, Graduate School of Business. https://ideas.repec.org/p/ecl/stabus/4220.html

[20] Daniel Halpern, Joseph Y. Halpern, Ali Jadbabaie, Elchanan Mossel, Ariel D. Procaccia, and Manon Revel. 2023. In Defense of Liquid Democracy. In *Proceedings of the 24th ACM Conference on Economics and Computation* (London, United Kingdom) *(EC '23)*. Association for Computing Machinery, New York, NY, USA, 852. https://doi.org/10.1145/3580507.3597817

[21] Steven Hardt and Lia Carrari R. Lopes. 2015. Google Votes: A Liquid Democracy Experiment on a Corporate Social Network. https://api.semanticscholar.org/CorpusID:152929756

[22] Wassily Hoeffding. 1963. Probability Inequalities for Sums of Bounded Random Variables. *J. Amer. Statist. Assoc.* 58, 301 (1963), 13–30. http://www.jstor.org/stable/2282952

[23] Johannes Jensen, Victor von Wachter, and Omri Ross. 2021. How Decentralized is the Governance of Blockchain-based Finance: Empirical Evidence from four Governance Token Distributions. https://doi.org/10.48550/arXiv.2102.10096

[24] Anson Kahng, Simon Mackenzie, and Ariel D. Procaccia. 2021. Liquid Democracy: An Algorithmic Perspective. *J. Artif. Intell. Res.* 70 (2021), 1223–1252.

[25] Christoph Kling, Jérôme Kunegis, Heinrich Hartmann, Markus Strohmaier, and Steffen Staab. 2021. Voting Behaviour and Power in Online Democracy: A Study of LiquidFeedback in Germany's Pirate Party. *Proceedings of the International AAAI Conference on Web and Social Media* 9, 1 (Aug. 2021), 208–217. https://doi.org/10.1609/icwsm.v9i1.14618

[26] Nathan Linial. 1987. Distributive graph algorithms Global solutions from local data. In *28th Annual Symposium on Foundations of Computer Science (sfcs 1987)*. 331–335. https://doi.org/10.1109/SFCS.1987.20

[27] Grace Liu. 2024. *The Illusion of Democracy— Why Voting in Decentralized Autonomous Organizations Is Doomed to Fail.* Research Papers 24-13. NYU School of Law. https://ssrn.com/abstract=4441178

[28] James C. Miller. 1969. A program for direct and proxy voting in the legislative process. *Public Choice* 7 (1969), 107–113. https://api.semanticscholar.org/CorpusID:154387226

[29] B. Pablo Montagnes, Zachary Peskowitz, and Suhas A. Sridharan. 2024. How Well Do Voting Choice Policies Represent Public and Investor Preferences?

[30] David Peleg. 2000. *Distributed computing: a locality-sensitive approach.* Society for Industrial and Applied Mathematics, USA.

[31] Stefan Schmid and Dmitry Shestakov. 2024. Invited Paper: Blockchain Governance and Liquid Democracy – Quantifying Decentralization in Gitcoin and Internet Computer. In *Proceedings of the 2024 Workshop on Advanced Tools, Programming Languages, and PLatforms for Implementing and Evaluating Algorithms for Distributed Systems* (Nantes, France) *(ApPLIED'24)*. Association for Computing Machinery, New York, NY, USA, 1–7. https://doi.org/10.1145/3663338.3663678

[32] Gordon Tullock. 1992. Computerizing politics. *Math. Comput. Model.* 16, 8–9 (Aug. 1992), 59–65. https://doi.org/10.1016/0895-7177(92)90087-2

[33] Chiara Valsangiacomo. 2021. Political Representation in Liquid Democracy. In *Frontiers in Political Science.* https://api.semanticscholar.org/CorpusID:232342352

[34] Shaileshh Bojja Venkatakrishnan, Giulia Fanti, and Pramod Viswanath. 2017. Dandelion: Redesigning the Bitcoin Network for Anonymity. *Proc. ACM Meas. Anal. Comput. Syst.* 1, 1 (2017), 22:1–22:34.

[35] Yuzhe Zhang and Davide Grossi. 2021. Power in Liquid Democracy. In *AAAI.* AAAI Press, 5822–5830.

A ADDITIONAL DEFINITIONS

A.1 Positive gain

Definition 6. *(Positive gain.) A mechanism M and set of graph properties $\mathcal{P}$ satisfies the positive gain property if there exists $\gamma > 0$ and $n' \in \mathbb{N}$ such that for all $n > n'$, there is a problem instance $G \in \mathcal{G}_n^{\mathcal{P}}$ where we have $gain(M, G) > \gamma$.*

Definition 7. *Let $x_1, x_2, \ldots$ be a sequence of independent Bernoulli random variables taking values in $\{0, 1\}$. For each x_i, let p_i denote the probability that x_i takes the value of 1. Let X_i be the random variable $\sum_{i' \leq i} x_{i'}$, and $\mu(X_i)$ be the expected value of X_i.*

B OMITTED PROOFS

B.1 Proof of Lemma 5

Lemma 5. *For every $\varepsilon > 0$ and every j, with probability at most $e^{-\Omega(j^{1/3})}$, there exists $i \geq j$ such that*

$$X_i < \left(1 - \frac{\varepsilon}{j^{1/3}}\right) \mu(X_i) .$$

Proof. Let us examine X_i for $i \leq j$. From the Chernoff bound, we have

$$P[X_i < \left(1 - \frac{\varepsilon}{j^{1/3}}\right) \mu(X_i)] \leq e^{-\Omega(i^{1/3})},$$

for a constant ε and $\mu(X_i) \in \Theta(i)$.

Now we look at the case where $i > j$. We can sum the probabilities where $X_i < \left(1 - \frac{\varepsilon}{j^{1/3}}\right) \mu(X_i)$ over all $i > j$. to get

$$\sum_{i>j} e^{-\Omega(i^{1/3})} \in e^{-\Omega(j^{1/3})},$$

which completes the proof. □

B.2 Proof of Lemma 1

Lemma 1. *Let X_n be a random variable associated with G, (j, c, n)-recycle-sampled graph, then we have with probability at least* $1 - e^{-\Omega(j^{1/3})}$

$$X_n \geq \mu(X_n) - c\frac{\varepsilon \cdot n}{j^{1/3}}.$$

Proof. **Step 1: (Bounding the failure probability of the first partition.)** Let c denote the partition complexity of M and let $\pi_1, \ldots, \pi_c$ denote each partition index. Let us consider the random variables that belong in the first partition. Recall that these would be the first π_1 elements of M, i.e., $\{x_1, \ldots, x_{\pi_1}\}$. Since this is the first partition, the random variables $\{x_1, \ldots, x_{\pi_1}\}$ are independent, hence we can use Lemma 5 to bound the probability that $X_{\pi_1} \leq \left(1 - \frac{\varepsilon}{j^{1/3}}\right) \mu(X_{\pi_1})$ to be $1 - e^{-\Omega(j^{1/3})}$ for some $j \leq \pi_1$.

Step 2: (Creating a modified sequence of independent random variables.) The random variables in M are recycle-sampled, hence dependent. We now define a new modified sequence $\hat{M} = \{\hat{x}_1, \ldots \hat{x}_n\}$ of *independent* Bernoulli variables, where we modify the Bernoulli parameter of random variables in the original sequence M if their outcomes depend on another random variable earlier in the sequence (i.e., with smaller index). Since the random variables in the first partition are already independent and, by definition of the first partition, the variables in the first partition cannot depend on variables in earlier partitions, $\hat{x}_1, \ldots, \hat{x}_{\pi_1} = x_1, \ldots, x_{\pi_1}$.

Now let us examine the random variables in the second partition, i.e., x_i for $\pi_1 < i \leq \pi_2$. Each m_i for $\pi_1 < i \leq \pi_2$ is created by the following thought experiment: suppose that for some $j < i' \leq \pi_1, X_{i'} < \left(1 - \frac{\varepsilon}{j^{1/3}}\right) \mu(X_{i'})$, i.e., the sum of correct votes of some subsequence in the first partition drops significantly below the mean of the subsequence. Such an event happens with probability $e^{-\Omega(j^{1/3})}$. When such an event occurs, this could affect the variables in the second partition that are dependent on the "bad outcomes" of the variables in the first partition, which could then result in $X_{\pi_2} < \left(1 - \frac{\varepsilon}{j^{1/3}}\right) \mu(X_{\pi_2})$ for some $j < \pi_2$. To avoid this, we pretend all outcomes of the random variables in the first partition are such that no such bad event happens, and create the modified random variable $\hat{x}_i$ for $\pi_1 < i \leq \pi_2$ where the Bernoulli parameter of $\hat{x}_i$ is modified to be $\frac{1}{i}\left(1 - \frac{\varepsilon}{j^{1/3}}\right) \mu(X_i)$ (whereas x_i has success probability $\frac{1}{i}\mu(X_i)$). In doing so, all variables in the second partition are independent of the variables in the first partition and their probability is decreased by $\frac{\varepsilon}{j^{1/3}}$. In other words, for $\pi_1 < i \leq \pi_2$, $\hat{x}_i \geq x_i - \frac{\varepsilon}{j^{1/3}}$.

Step 3: (Applying the previous steps recursively for all partitions.) For a general partition $2 \leq t \leq c$, we apply the modification procedure described in the previous step to the variables in the partition. Suppose the following inequality holds for all variables in previous partitions:

$\hat{x}_i \geq x_i - D$. Let $\hat{X}_{\pi_t} = \Sigma_{i=1}^{\pi_t} \hat{x}_i$. Then, for all $i \leq \pi_t$ all the variables $\{\hat{x}_1, \ldots, \hat{x}_{\pi_t}\}$ are independent, and we can apply Lemma 5 to get

$$\mu(\hat{X}_i) \geq (\mu(X_i) - D)\left(1 - \frac{\varepsilon}{j^{1/3}}\right),$$

with probability at most $e^{-\Omega(j^{1/3})}$ for $j < i$. The RHS of the inequality is asymptotically $\mu(X_i) - \left(D + \frac{\varepsilon n}{j^{1/3}}\right)$. This means that the success probability for a variable in partition t is at least $\hat{x}_i \geq x_i - \left(D + \frac{\varepsilon n}{j^{1/3}}\right)$. Solving the recurrence, we get that the decrease in probability is at most $(t-1)\frac{\varepsilon}{j^{1/3}}$ in partition t.

Now, we can again use the Chernoff bound to show that with probability at least $1 - e^{-\Omega(j^{1/3})}$ we have

$$\hat{X}_n \geq \mu(X_n) - c\frac{\varepsilon \cdot n}{j^{1/3}}.$$

□

B.3 Proof of Lemma 2

We first state a lemma as written in [24] which shows and proves the convergence to a normal distribution of the sum of independently distributed Bernoulli random variables. This corresponds to the distribution of the total number of correct votes of direct voting in the bounded competencies setting.

Lemma 6 (from Kahng et al. [24]). *Let $Y = \{Y_1, Y_2, \ldots, Y_n, \ldots\}$ be a sequence of independent Bernoulli random variables where Y_i has success probability $p_i \in [\beta, 1-\beta]$ for $\beta \in (0, 1/2)$. Then $\sum_{k=1}^{n} Y_k$ converges to a normal distribution with mean $\sum_{k=1}^{n} E[Y_k]$ and variance $\sum_{k=1}^{n} Var[Y_k]$ as n goes to infinity; i.e.,*

$$\sum_{k=1}^{n} Y_k \longrightarrow N\left(\sum_{k=1}^{n} E[Y_k], \sum_{k=1}^{n} Var[Y_k]\right) \text{ as } n \longrightarrow \infty.$$

Now, we are ready to prove Lemma 2 itself.

Lemma 2. *For all $\varepsilon > 0$ and all $\mathcal{P}$ where $(\mathbf{p} \in (\beta, 1-\beta)) \in \mathcal{P}$, any mechanism that delegates less than $n^{\frac{1}{2}-\varepsilon}$ votes satisfies DNH.*

Proof. The variance of a single Bernoulli random variable with parameter that lies in the interval $[\beta, 1-\beta]$ for $\beta \in (1, \frac{1}{2})$ lies in the interval $[\beta(1-\beta), \frac{1}{4}]$, since the variance increases with β, up to a maximum of $\frac{1}{4}$. Using our notation, recall that X_n^D denotes the sum of the total number of correct votes for the direct voting mechanism. Since each variable is independent, from Lemma 6 we have that X_n^D is a random variable that approaches the normal distribution with mean $\mu(X_n^D)$ and the variance in $[n\beta(1-\beta), \frac{n}{4}]$. That means the standard deviation σ of X_n^D must lie in $[\sqrt{n\beta(1-\beta)}, \frac{\sqrt{n}}{2}]$.

Now suppose that the $n^{\frac{1}{2}-\varepsilon}$ number of voters that delegated their votes in the delegated setting all voted incorrectly (i.e., the voters they delegated to voted incorrectly, hence their votes are counted as incorrect as well). Let us also suppose that in the direct voting setting all of these voters (the ones who delegated their votes as well as the ones who got votes delegated to) all voted correctly. This means that the loss of the delegated mechanism where at most $n^{\frac{1}{2}-\varepsilon}$ delegate is in the worst case $2n^{\frac{1}{2}-\varepsilon}$.

We need to compute the probability that the outcome of voting changes due to delegation. This means, in the direct voting setting, we have over $\frac{1}{2}n$ correct votes and in the delegated setting, we have at least $\frac{1}{2}n$ incorrect votes.

We want to compute the probability that X_n^D lies in $(\mu(X_n^D) - 2n^{\frac{1}{2}-\varepsilon}, \mu(X_n^D) + 2n^{\frac{1}{2}-\varepsilon})$. Since the distribution of X_n^D for large n approaches the normal distribution with standard deviation of order $\sqrt{n}$, we can use the error function for the normal distribution to bound this probability.

For σ, the standard deviation, this probability is at most

$$\operatorname{erf}\left(\frac{2n^{\frac{1}{2}-\varepsilon}}{\sigma}\right) \leq \operatorname{erf}\left(\frac{n^{\frac{1}{2}-\varepsilon}}{\sqrt{2}n^{\frac{1}{2}}}\right) = \operatorname{erf}\left(\frac{n^{-\varepsilon}}{\sqrt{2}}\right)$$

But we know

$$\lim_{n\to\infty} \operatorname{erf}\left(\frac{n^{-\varepsilon}}{\sqrt{2}}\right) = 0\,.$$

This means the probability that the outcome is changed in the delegated setting compared to direct voting goes to 0 asymptotically. □

B.4 Proof of Lemma 3

First, we state Hoeffding's inequality [22], then use it to prove Lemma 7.

Theorem 5 (Hoeffding's inequality). *Let $X_1, X_2, \ldots, X_n$ be independent random variables such that $a_i \leq X_i \leq b_i$, then for $S = \sum_{i=1}^n X_i$ holds*

$$P[|S - E[S]| \geq t] \leq 2\exp\left(-\frac{2t^2}{\sum_{i=1}^n (b_i - a_i)^2}\right)\,.$$

Lemma 7. *Let $x_1, x_2, x_3, \ldots, x_n$ be the Bernoulli random variables representing sinks in the delegation graph with weights $w_1, w_2, w_3 \ldots, w_n$. Let $\mu(X) := \sum_i w_i p_i$ and $X := \sum_i w_i x_i$.*

Let $w := \max_i w_i$. Then for any constant $c > 0$, we have

$$|X - \mu(X)| \leq \frac{1}{c}\sqrt{n^{1+\varepsilon}w}\,.$$

with probability $1 - e^{-\Omega(n^\varepsilon)}$.

Proof. The biggest weight is w, so there has to be at least $\frac{n}{w}$ sinks. We plug into Theorem 5, we have $(b_i - a_i)^2 \leq w^2$ and this gives at least $\frac{n}{w}$ summands. Then we have

$$P[|X - \mu(X)| \geq t] \leq 2\exp\left(-\frac{2t^2}{nw}\right)\,.$$

setting $t = \sqrt{n^{1+\varepsilon}w}$, we get

$$P[|X - \mu(X)| \geq \sqrt{n^{1+\varepsilon}w}] \leq 2\exp\left(-n^\varepsilon\right)\,,$$

which completes the proof. □

Lemma 3. *Let X_n be the voting outcome any delegation mechanism where every sink has weight at most w. Then with probability $1 - e^{-\Omega(n^\varepsilon)}$ for any constant $c > 0$ we have*

$$|\mu(X_n) - X_n| \leq \frac{1}{c}\sqrt{n^{1+\varepsilon}w}\,.$$

Proof. Simple instantiation of Lemma 7. □

B.5 Proof of Lemma 4

Lemma 4. *(Increase in expectation.) Let Y_n be sequence given by Algorithm 1 where k voters do not delegate, we have*

$$P[Y > \mu(X_n) + (n-k)\alpha - \frac{1}{\alpha}\frac{\varepsilon n}{j(n)^{1/3}}] > 1 - e^{-\Omega(j(n)^{1/3})}\,.$$

Proof. Let V_k denote the set of voters that did not delegate their vote. Since $j(n)$ serves as the threshold determining if a voter should delegate their vote or not in Algorithm 1, we know that $k \geq j(n)$. From this we know that for all of the voters that delegated, the expected value of their delegated vote is an average of all approved neighbours. This means the random variables in Y_n create $(j(n), n)$-recycle-sampled random variables and the expected value of the outcome y_i of voter v_i that delegated their vote is at least $p_i + \alpha$ (since all of v_i's approved neighbours are at least α more competent than v_i).

Since K_n is complete, the partition complexity of Y_n is $\frac{1}{\alpha}$. We can split the interval $[0,1]$ into partitions of size α with the ith partition contains all voters with competency level between $[(i-1)\cdot\alpha, i\cdot\alpha]$ for $i \in [\frac{1}{\alpha}]$. From our definition of partitions, any voter lying in any one of these partitions cannot approve of anyone else inside the same partition. That means we can use Lemma 1 to show that with probability $1 - e^{-\Omega(j(n)^{1/3})}$, we have

$$Y \geq \mu(Y) - \frac{1}{\alpha}\frac{\varepsilon n}{j(n)^{1/3}} \tag{1}$$

Now, we need to compute $\mu(Y)$, the expected value of the sum of the correct votes of the sequence Y_n. We first observe that every voter that delegated their vote increased the expected total sum $\mu(Y)$ by at least α. This gives us the following lower bound for $\mu(Y)$:

$$\mu(Y) \geq \mu(X_n) + (n-k)\alpha\,.$$

Combining this with the previous bound in Equation (1), we get

$$P[Y > \mu(X_n) + (n-k)\alpha - \frac{1}{\alpha}\frac{\varepsilon n}{j(n)^{1/3}}] < e^{-\Omega(j(n)^{1/3})}\,.$$

□

B.6 Proof of Theorem 1

Theorem 1. *Algorithm 1 satisfies*

- *SPG for graph properties $\mathcal{P} = \{K_n, \mathrm{PC} = \frac{\alpha}{k}\}$ and* $\mathrm{Delegate}(n) > \frac{n}{k+\varepsilon}$ *for some $k > 1$ and $\varepsilon > 0$,*
- *and DNH for $\mathcal{P} = \{K_n\}$.*

Proof. From Lemma 4, we have a positive gain for $\Omega(\frac{n}{j(n)^{1/3}})$ delegated votes. From Lemma 2, we have a DNH for competencies in $(\beta, 1-\beta)$ for $O(n^{1/2-\varepsilon})$ delegated votes. We note that the proof of Lemma 2 requires the standard deviation of the sequence of independent Bernoulli random variables (corresponding to the voting outcomes in direct voting) to be at least $\Omega(\sqrt{n})$. This means that the requirement for the competency to be in $(\beta, 1-\beta)$ can be violated for some constant say $\frac{9}{10}n$ voters and the theorem still holds, since in this case the standard deviation is scaled only by $\frac{1}{10}$.

Now, suppose that at least $\frac{9}{10}n$ of voters have competencies outside the interval $(\beta, 1-\beta)$. We select $\beta < \min(\frac{1-\alpha}{2}, \frac{1}{7}\alpha)$ (since $0 < \alpha < 1$).

Recall that in the complete graph setting, voters delegate only if they have more than $j(n)$ approved neighbours. Now we split the distribution of the $\frac{9}{10}n$ voters that lie outside the interval into two cases:

(1) there are at least $j(n)$ voters with competency larger than $1-\beta$

(2) less than $j(n)$ voters have competency larger than $1-\beta$

Let us analyse the first case. If at least $j(n)$ voters have competency above $1-\beta$, we know all voters with competency below β delegate. This is due to the fact that β is chosen such that $1-2\beta > \alpha$, and since the graph is complete, so every voter sees the top $j(n)$ voters with competencies $> 1-\beta$ and delegates to them. Now let us examine the total number of vote delegations. From Lemma 4 we know that if at least $\frac{1}{10}n$ voters delegate we have positive gain. When less than $\frac{1}{10}$ voters delegate, we know that at least $\frac{8}{10}n$ voters have competency above $1-\beta$. Recall we chose β such that $\beta < \min(\frac{1-\alpha}{2}, \frac{1}{7}\alpha)$, which means that $\beta < 1/3$. Since $\beta < 1/3$, the expected number of correct votes would be at least $\frac{8}{10}n \cdot \frac{2}{3} = \frac{8}{15}n > \frac{1}{2}n$. This means that for this case where at least $j(n)$ voters have competency above $1-\beta$ these highly competent voters already ensure the voting outcome will be correct with high probability.

Now we analyse the second case where less than $j(n)$ voters have competency above $1-\beta$. In this setting, we have $\frac{9}{10}n - j(n)$ voters with competency at most β. The $\frac{1}{10}n$ voters with competencies in the interval $(\beta, 1-\beta)$ thus have competency at most $1-\beta$, and the remaining less than $j(n)$ voters have competency at most 1. Recall we assume that $j(n) < \frac{1}{3}n$. Hence, the expected number of correct votes is at most

$$\frac{1}{3}n + \frac{17}{30}\beta n + \frac{3}{30}(1-\beta)n\,,$$

which is strictly less than $\frac{1}{2}n$ for $\beta < \frac{1}{7}$. That means that the voting outcome is incorrect with a high probability. □

B.7 Proof of Theorem 2

The important observation is that the output of Algorithm 1 with threshold $j(d)$ and output of Algorithm 2 with threshold $n\frac{j(d)}{d}$ on the same input behave very similarly. The only difference is that the Algorithm 1 delegates surely, whereas Algorithm 2 delegates in expectation.

Lemma 8. *Let Y_n be sequence given by Algorithm 2 where k voters do not delegate. We have*

$$P[Y > \mu(X_n) + (n-k)\alpha - \frac{2}{\alpha}\frac{\varepsilon n}{n^{\varepsilon'}}] > 1 - e^{-\Omega\left(n^{\varepsilon'}\right)}\,.$$

Proof. If $j(d) \in \Omega(n^{\varepsilon'})$, then we have $(j(d), \frac{1}{\alpha}, n)$ recycled random variables. The first $j(d)$ voters cannot delegate and all others have, before sampling, the same probability to delegate to all voters that are in the approved set (if they delegate). Then the proof follows.

Otherwise, we examine the first $\frac{j(d)}{d}n^{\varepsilon'}$ voters. They delegate with probability at most

$$\sum_{i \geq j(d)} \binom{d}{i} \left(\frac{1}{n^{1-\varepsilon}}\right)^i .$$

If any voter among the first $\frac{j(d)}{d}n^{\varepsilon'}$ voters delegates, we treat it instead as voting incorrectly. That means every voter among the first decreases its p_i by at most $n^{-1+\varepsilon'}$. This creates $(n^{\varepsilon'}, \frac{1}{\alpha}, n)$-recycled random variables. However, the delegation does not produce an increase of correctness probability by α, but only by $\alpha - n^{-1+\varepsilon'}$. Hence the decrease of the bound by $\frac{2}{\alpha}\frac{\varepsilon n}{n^{\varepsilon'}}$. □

Theorem 2. *Algorithm 2 satisfies*

- *SPG for graph properties $\mathcal{P} = \{\mathrm{Rand}(n,d), \mathrm{PC} = \frac{\alpha}{k}\}$ and $\mathrm{Delegate}(n) > \frac{n}{k+\varepsilon}$ for some $k > 1$ and $\varepsilon > 0$,*
- *and DNH for graph property $\mathcal{P} = \{\mathrm{Rand}(n,d)\}$.*

Proof. Proof of SPG follows from Lemma 8.

We prove the DNH property for the random d-regular graph as a small change to the proof of the DNH property for the complete graph. We can replace everywhere $j(n)$ by $\frac{j(d)}{d}n$ and all certainties are now happening in expectation. Since these expectations are concentrated around the mean, we can argue about them as in the complete graph. □

B.8 Proof of Theorem 3

Theorem 3. *Let $\mathcal{M}$ be any delegation mechanism, then $\mathcal{M}$ satisfies*

- *SPG for properties $\mathcal{P} = \{\Delta \le t^{\alpha/(1+\varepsilon)}, \mathrm{PC} = \frac{t\alpha}{2n}\}$ and $\mathrm{Delegate}(n) > t$ for any t and $\varepsilon > 0$,*
- *and DNH for properties $\mathcal{P} = \{\Delta \le n^{\alpha/(2+\varepsilon)}, \mathbf{p} \in (\beta, 1-\beta)\}$.*

Proof. For SPG, we first show the increase in the expected value, and then the statement of the theorem is a direct consequence. We only examine voters who participated in the liquid democracy, which means they either delegated their vote or were delegated to. We denote them $X = \{x_1, x_2, \ldots, x_n\}$ and X' after delegating (note that $2t \ge n$). We show that with high probability, the result of delegating is higher than the direct voting.

We have $\mu(X)$ the expectation of X. From the Chernoff bound, we have, that with probability $1 - e^{-O(t^{1/3})}$, the value of X is below $\mu(X)(1 + t^{-1/3})$. Observe that $\mu(X') \ge \mu(X) + t\alpha$, because every voter who delegated increases their probability by at least α.

Let us compute the maximum weight that one voter can have with respect to the maximal degree Δ. It can get votes from at most Δ other voters, which can get votes from $\Delta - 1$ other neighbors. The longest delegation path is $\frac{1}{\alpha}$, which implies the maximum weight is $\Delta^{\frac{1}{\alpha}}$.

From Lemma 7, with probability $1 - e^{-\Omega(n^{\varepsilon})}$, for constant $\frac{\alpha}{4}$ we have

$$|\mu(X') - X'| \le \frac{\alpha}{4}\sqrt{n^{1+\varepsilon}\Delta^{\frac{1}{\alpha}}} \le \frac{\alpha}{4}\left((2t)^{1+\varepsilon}\left(t^{\alpha/(1+\varepsilon)}\right)^{\frac{1}{\alpha}}\right)^{1/2} \le \frac{\alpha}{2}t\,,$$

which means with a high probability the value of X' is above $\mu(X) + \frac{\alpha}{2}t$ which with $\mathrm{PC} = \frac{t\alpha}{2}$ gives a strong positive gain.

For DNH, suppose that only $n^{\frac{1}{2}-\varepsilon}$ voters delegate. DNH then follows as a consequence of Lemma 2. If more voters delegate, then the instance satisfies all properties of Theorem 3 and we guarantee the increase of expectation. □

B.9 Proof of Theorem 4

Theorem 4. *Let $\mathcal{M}$ be a delegation mechanism where a voter delegates if at least $\frac{2}{3}$ of its neighbors are approved. Then, $\mathcal{M}$ satisfies*

- *SPG for properties $\mathcal{P} = \{\delta \ge n^{\varepsilon}, \mathrm{PC} = \frac{\alpha h}{2n}\}$ and $\mathrm{Delegate}(n) > h$ for any $h \ge \sqrt{n}$ and $\varepsilon > 0$,*
- *and DNH for properties $\mathcal{P} = \{\delta \ge n^{\varepsilon}, \mathbf{p} \in (\beta, 1-\beta)\}$.*

Proof. We first examine only the voters who delegated their vote, let there be t of them. We show an increase in expectation in the outcome of these voters. Note that the expectation of the outcome of the voters who did not delegate is independent of the above increase. We use Lemma 3, for this, we bound the maximum weight of v, the most competent voter. We describe a construction that maximizes the probability for every voter to delegate to v. We split the voters into sets, the first set contains at least $\frac{2}{3}\delta$ voters with the highest competencies, and the next sets contain the voters with competencies within width α (i.e. let v_i be the voter with competency p_i who is not in a set, then we create another set that contains all voters with competencies between p_i and $p_i - \alpha$). We index the sets by $s_0, s_1, \ldots$. Observe that voters can delegate only to a set with a lower index. We bound the maximal density of delegation to v for each set of voters.

The expected density in s_1 is at most $\frac{3}{2\delta}$ since every voter that delegates has an approved set of size at least $\frac{2}{3}\delta$ and only one element of it can be v. From Chernoff bound, with probability at least $1 - e^{-n^\varepsilon}$, we have the density at most $\frac{3}{\delta}$.

The expected density in s_2 is at most $\frac{3}{\delta} + \frac{3}{2\delta} = \frac{9}{2\delta}$. This is because one edge out of $\frac{2}{3}\delta$ can lead to v, and the rest of the edges lead to s_1, where the density is $\frac{3}{\delta}$. Note that since at least $\frac{2}{3}$ of neighbors need to be approved, the number of edges for every voter in s_1 between s_2 and s_1 can be at most $\frac{1}{2}$ of the number of edges between s_1 and s_0. Also, if there are some vertices in s_1 that do not have edges to s_2, we can move them to s_2 and increase the density that way. This means that the variance cannot accumulate. Again, from Chernoff bound, with probability at least $1 - e^{-n^\varepsilon}$, we have the density at most $\frac{9}{\delta}$.

Given d_i, the maximal density in s_i, we compute the maximal density in s_{i+1}. The density is at most $d_i + \frac{3}{2}\delta$, since an edge can lead to v and all other edges from s_{i+1} to s_i. From Chernoff bound, with probability at least $1 - e^{-n^\varepsilon}$, we have the density at most $2d_i + \frac{3}{\delta}$.

Solving for i, we get that the density in s_i is at most $\frac{3^i}{\delta}$. This means that the maximal density is at most $\frac{3^{1/\alpha}}{\delta}$ with probability at least $1 - \frac{1}{\alpha}e^{-n^\varepsilon}$. Therefore, with probability at least $1 - \frac{1}{\alpha}e^{-n^\varepsilon}$, at most $3^{1/\alpha}\frac{1}{\delta}n$ voters delegated to v.

Let X be the random variable denoting the result of direct voting of all t voters who delegated. Let X' be the random variable after delegation. Then from Chernoff bound, with probability at most $1 - e^{-\Omega(t)}$, the value is below $E[X] + \frac{\alpha t}{2}$.

Every act of delegation increases the expectation by at least α, which means $\mu(X') \geq \mu(X) + \alpha \cdot t$. We know that the maximal weight is of v and it is at most $3^{1/\alpha}\frac{1}{\delta}t$. This allows us to use Lemma 3, for $n = t$ and $w = 3^{1/\alpha}\frac{1}{\delta}t$. Therefore, we have with probability $1 - e^{-\Omega(t^{\varepsilon'})}$ the difference is

$$|\mu(X') - X'| \leq \frac{1}{c}t\sqrt{t^{\varepsilon'}3^{1/\alpha}\delta^{-1}}\,.$$

Since $\delta \geq n^\varepsilon \geq t^\varepsilon$, we have $\frac{1}{c}t\sqrt{t^{\varepsilon'}3^{1/\alpha}\delta^{-1}} \leq \frac{\alpha}{2}t$ for some ε'.

Having $t \geq \sqrt{n}$ ensures the increase of the expectation by $\alpha t/2$, so for $\mathrm{PC} = \frac{\alpha t}{2n}$, we have a strong positive gain.

Finally, DNH is a consequence of SPG for many delegated votes and Lemma 2 for a small number of delegated votes. □