

Weighted Packet Selection for Rechargeable Links: Complexity and Approximation

Stefan Schmid

Technische Universität Berlin, Germany

Jakub Svoboda

Institute of Science and Technology, Austria

Michelle Yeo

Institute of Science and Technology, Austria

Abstract

We consider a natural problem dealing with weighted packet selection across a rechargeable link, which e.g., finds applications in cryptocurrency networks. The capacity of a link (u, v) is determined by how much players u and v allocate for this link. Specifically, the input is a finite ordered sequence of packets that arrive in both directions along a link. Given (u, v) and a packet of weight x going from u to v , player u can either accept or reject the packet. If player u accepts the packet, their capacity on link (u, v) decreases by x . Correspondingly, player v 's capacity on (u, v) increases by x . If a player rejects the packet, this will entail a cost linear in the weight of the packet. A link is "rechargeable" in the sense that the total capacity of the link has to remain constant, but the allocation of capacity at the ends of the link can depend arbitrarily on players' decisions. The goal is to minimise the sum of the capacity injected into the link and the cost of rejecting packets. We show the problem is NP-hard, but can be approximated efficiently with a ratio of $(1 + \varepsilon) \cdot (1 + \sqrt{3})$ for some arbitrary $\varepsilon > 0$.

2012 ACM Subject Classification Theory of computation \rightarrow Theory and algorithms for application domains; Theory of computation \rightarrow Approximation algorithms analysis; Networks \rightarrow Network algorithms

Keywords and phrases network algorithms, approximation algorithms, complexity, cryptocurrencies

Funding *Stefan Schmid*: Research supported by the European Research Council (ERC), grant agreement No. 864228 (AdjustNet), Horizon 2020, 2020-2025

1 Introduction

This paper considers a novel and natural throughput optimization problem where the goal is to maximise the number of packets routed through a network. The problem variant comes with a twist: link capacities are "rechargeable", which is motivated, e.g., by payment-channel networks (more motivation will follow).

We confine ourselves to a single capacitated network link and consider a finite ordered sequence of packet arrivals in both directions along the link. This can be modelled by a graph that consists of a single edge between two vertices u and v , where b_u and b_v represent the capacity u and v injects into the edge respectively. Each packet in the sequence has a weight/value and a direction (either going from u to v , or from v to u). When u forwards a packet going in the direction u to v , u 's capacity b_u decreases by the packet weight and v 's capacity b_v correspondingly increases by the packet weight (see Figure 1 for an example). Player u can also reject to forward a packet, incurring a cost linear in the weight of the packet. The links we consider are rechargeable in the sense that the total capacity $b_u + b_v$ of the link can be arbitrarily distributed on both ends, but the total capacity of the link cannot be altered throughout the lifetime of the link. Given a packet sequence, our goal is to minimise the sum of the cost of rejecting packets and the amount of capacity allocated to a

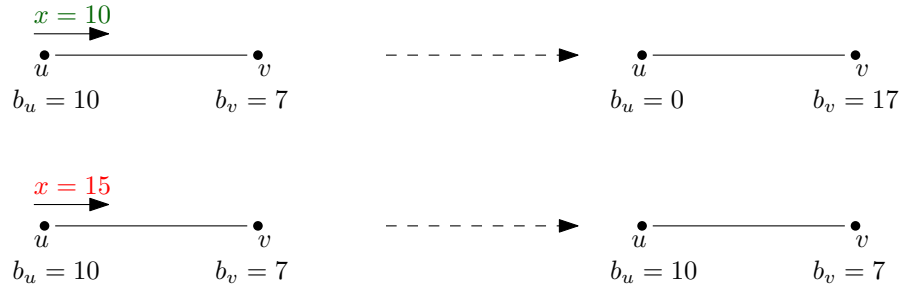
link.

Here we stress a crucial difference between our problem and problems on optimising flows and throughput in typical capacitated communication networks [5, 13]. In traditional communication networks, the capacity is usually independent in the two directions of the link [9]. In our case, however, the amount of packets u sends to v in a link (u, v) directly affects v 's capability to send packets, as each packet u send to v increases v 's capacity on (u, v) .

We start with a description of rechargeable links, then explain the actions players can take and corresponding costs. We subsequently motivate and explain our problem with a real world example of routing payments in cryptocurrency networks. Finally, we state our main results.

Rechargeable links. One unique aspect of our problem is that the links we consider are rechargeable. Rechargeable links are links that satisfy the following properties:

1. Given a link (u, v) with total capacity M , the capacity can be arbitrarily split between both ends based on the number and weight of packets processed by u and v . That is, b_u and b_v can be arbitrary as long as $b_u + b_v = M$.
2. The total capacity of a link is invariant throughout the lifetime of the link. That is, it is impossible for players to add to or remove any part of the capacity in the link. In particular, if a player is incident to more than one link in the network, the player cannot transfer part of their excess capacity in one link to "top up" the capacity in another one.



■ **Figure 1** The diagram on the top shows the outcome of u successfully processing a packet x of weight 10 along the link (u, v) . The subsequent capacities of u and v are 0 and 17 respectively. The diagram on the bottom shows the outcome where, even though the total capacity of the (u, v) link is 17, u 's capacity of 10 on (u, v) is insufficient to forward a packet x of weight 15. As such, the subsequent capacities of u and v on the link (u, v) remain the same.

Player actions and costs. First, we note that creating a link incurs an initial cost of the amount the player allocates in the link. That is, if player u allocates b_u in link (u, v) , the cost of creating the link (u, v) for u would be b_u . Consider a link (u, v) in the network and a packet going from u to v along the edge. Player u can choose to do the following to the packet:

- **Accept packet.** Player u can accept to forward the packet if their capacity in (u, v) is at least the weight of the packet. The result of doing so decreases their capacity by the packet weight and increases the capacity of v by the packet weight. Note that apart from gradually depleting a player's capacity, accepting the packet does not incur any cost.

- **Reject packet.** Player u can also reject the packet. This could happen if u 's capacity is insufficient, or if accepting the packet would incur a larger cost in the future. For a packet of weight x , the cost of rejecting the packet is $f \cdot x + m$ where $f, m \in \mathbb{R}^+$.

We note that player u does not need to take any action for packets going in the opposite direction (i.e. from v to u) as these packets do not add anything to the cost of u . See Section 2 for the formal details of packets.

Practical motivation. The primary motivating example of our model is payment channel networks [7, 8] supporting cryptocurrencies [1, 12]. These networks are used to route payments of some amount (i.e. weighted packets in our model) between any two users of the network. Channels (i.e. links in our model) are capacitated, which can limit transaction throughput and hence may require selection and recharging [4, 10]. Intermediate nodes on a payment route typically also charge a fee for forwarding payments that is linear in the amount of the payment. Hence, if they reject to forward a payment, they would lose out on profiting from this fee and thereby incur the fee amount as opportunity cost. Channels in payment channel networks are also rechargeable for security reasons, see [12] for more details.

Our contributions. We introduce the natural weighted packet selection problem and show that it is NP-hard by a reduction from subset sum. Our main contribution is an efficient constant-factor approximation algorithm. We further initiate the discussion of how our approach can be generalized from a single link to a more complex network.

Organisation. Section 2 introduces the requisite notations and definitions we use in our paper. Section 3 provides the necessary algorithmic building blocks we use to construct our main algorithm. In Section 4, we present our main approximation algorithm and prove that it achieves an approximation ratio of $(1 + \varepsilon)(1 + \sqrt{3})$ for the weighted packet selection for a link problem in Theorem 10. We show that weighted packet selection for a link is NP-hard in Section 5. Finally, we provide some generalisations of our algorithm from a single link to a larger network in Section 6. We conclude our work by discussing future directions in Section 7.

2 Notation and definitions

Let (u, v) be link. We denote an ordered sequence of packets by $X_t = (x_1, \dots, x_t)$. Each packet $x_i \in X_t$ has a weight and a direction. We simply use $x_i \in \mathbb{R}^+$ to denote the weight of the packet x_i . We say a packet x_i goes in the left to right direction (resp. right to left) if it goes from u to v (resp. from v to u).

Let X_{\rightarrow} denote the subsequence of X_t that consist of packets going from left to right and X_{\leftarrow} the subsequence of X_t that consist of packets going from right to left.

Let x_{\min} be the weight of the packet with the smallest weight in X_t and M_{\max} be the capacity needed to accept all packets. The value of M_{\max} for X_t is easy to compute in time $\mathcal{O}(t)$ and it is upper bounded by the sum of the weight of all packets.

Let OPT be the cost of the optimal algorithm and OPT_M be the cost of the optimal algorithm using a capacity of M in the link. Moreover, we use OPT^R to denote the cost of the optimum for rejecting packets and OPT^C to denote the cost for the capacity. Similarly, we use OPT_M^R to denote the cost for rejecting of the optimum using capacity M in the link (note that $OPT_M^C = M$).

Finally, for an integer $t \geq 1$, we use $[t]$ to denote $\{1, \dots, t\}$.

3 Preliminary insights and algorithmic building blocks

We start our investigation of the weighted packet selection for a link problem by describing a procedure to approximate the optimal capacity in a link. We describe a linear program that fractionally accepts packets (i.e. part of a packet can be accepted) given a fixed link capacity M . Then, we introduce a simple algorithm that requires twice as much link capacity compared to the capacity required in the linear program. However, the algorithm guarantees that every packet accepted fully (i.e. the entire packet was accepted) by the linear program is also accepted by the algorithm. This controls the cost of the algorithm: the fully accepted packets do not increase the cost since the solution of the linear program is a lower bound on the optimal cost.

3.1 Approximating the optimal capacity

We present a lemma that allows us to fix the capacity to M for a small trade-off in the approximation ratio. We fix some $\varepsilon > 0$ and perform a search on all $k \in \mathbb{N}$ such that $x_{\min}(1 + \varepsilon)^k \leq M_{\max}$.

Observe that if the optimal capacity is not 0, it is at least x_{\min} , the weight of the smallest packet; and at most M_{\max} , the capacity needed to accept all packets.

► **Lemma 1.** *For any $\varepsilon > 0$, let $\mathcal{M} = \{x_{\min}(1 + \varepsilon)^k \mid k \in \mathbb{N} \text{ and } x_{\min}(1 + \varepsilon)^k \leq M_{\max}\} \cup \{0\}$. Then, the following inequality holds for any $LB_M \leq OPT_M^R$*

$$\min_{M \in \mathcal{M}} \left(LB_M + \frac{M}{1 + \varepsilon} \right) \leq OPT$$

Proof. If OPT rejected all packets, we know that $LB_0 \leq OPT_0^R = OPT_0 = OPT$, so the inequality holds.

Now, suppose that OPT accepted at least one packet. That means $OPT^C \geq x_{\min}$. So there exists k such that $x_{\min}(1 + \varepsilon)^{k-1} \leq OPT^C \leq x_{\min}(1 + \varepsilon)^k$. We set $M = x_s(1 + \varepsilon)^k$ and prove that $LB_M + \frac{M}{1 + \varepsilon} \leq OPT^R + OPT^C$. From the choice of M , we know that $\frac{M}{1 + \varepsilon} \leq OPT^C$.

Observe that for $M' \geq M$ holds $OPT_{M'}^R \leq OPT_M^R$. In the worst case, the same set of packets can be accepted with a larger capacity. And since $OPT^C \leq M$, it means $OPT^R \geq OPT_M^R \geq LB_M$. ◀

In Section 4, we describe an algorithm that is a $(1 + \sqrt{3})$ -approximation of LB_M . Together with Lemma 1, we can use this algorithm to approximate the whole problem with a ratio of $(1 + \varepsilon)(1 + \sqrt{3})$ by running the algorithm at most $\frac{1}{\varepsilon} \log \frac{M_{\max}}{x_{\min}}$ times. We note that choosing a smaller value of ε yields a better approximation, but increases the running time.

3.2 Linear program formulation

Here, we describe a linear program that computes a lower bound for OPT_M^R .

Observe that OPT_M rejects packets with weight larger than M . For the rest of the analysis, we assume that all packets in X_t have weight smaller than M .

In the linear program, we create a variable $0 \leq y_i \leq x_i$ for every packet $x_i \in X_t$ that represents the extent to which the packet is accepted ($y_i = \frac{x_i}{2}$ means that half of x_i is accepted). We introduce variables $S_{L,i}$ and $S_{R,i}$ denoting the capacity on the left and right ends of the link after processing first i packets from X_t . We know that $S_{L,i} + S_{R,i} = M$, and $0 \leq S_{L,i}, S_{R,i} \leq M$.

Now we can formulate the linear program in eq. (1):

$$\begin{aligned}
& \text{minimise} && \sum_i f(x_i - y_i) + m \frac{x_i - y_i}{x_i} && (1) \\
& \text{subject to} && \forall i : y_i, S_{L,i}, S_{R,i} \geq 0 \\
& && \forall i : y_i \leq x_i \\
& && \forall i : S_{L,i} + S_{R,i} = M \\
& && \forall x_i \in X_{\rightarrow} : S_{L,i} = S_{L,i-1} - y_i \\
& && \forall x_i \in X_{\rightarrow} : S_{R,i} = S_{R,i-1} + y_i \\
& && \forall x_i \in X_{\leftarrow} : S_{L,i} = S_{L,i-1} + y_i \\
& && \forall x_i \in X_{\leftarrow} : S_{R,i} = S_{R,i-1} - y_i
\end{aligned}$$

Let LP_M be the solution of the linear program with capacity parameter M . Lemma 2 states that LP_M is a lower bound of the optimal cost of the weighted packet selection for a link problem with capacity M .

► **Lemma 2.** *For all M , $OPT_M \geq LP_M$.*

Proof. The solution of OPT_M is an admissible solution to the linear program. If some other (fractional) solution is found, we know that it is at most OPT_M . ◀

The linear program can be solved in time $\mathcal{O}(n^\omega)$ where n is the number of variables in the linear program and ω the matrix multiplication exponent [6] (currently ω is around 2.37).

3.3 Processing fully accepted packets

Given the solution of the linear program with capacity M , we describe an algorithm that: uses capacity $2M$; accepts all packets that were fully accepted ($x_i = y_i$) by the linear program; and accepts some fractionally accepted packets.

Algorithm 1 describes the decision making process only for packets coming from left to right on the link, i.e. X_{\rightarrow} . The algorithm takes as input the solution to the linear program and the packet sequence X_t . Recall that $S_{L,i}$ and $S_{R,i}$ for $i \in [t]$ are the capacity distributions on the left and right end of the link after processing the i th packet from the linear program solution. First, we set R_L and R_R be $\frac{M}{2}$ each and we start with $S_{L,0} + R_L$ capacity on the left and $S_{R,0} + R_R$ capacity on the right of the link. Intuitively, one can think of the additional M capacity in R_L and R_R as an excess capacity to accept fractionally accepted packets or balance change in $S_{L,i}$ and $S_{R,i}$ in the case of packet rejection. We stress that the algorithm always uses the capacity in $S_{L,i}$ and $S_{R,i}$ to accept the fractional portion of the i th packet. The remainder capacity comes from R_L and R_R .

Since the problem is symmetric, one can simply swap $S_{L,i}$ for $S_{R,i}$, X_{\rightarrow} for X_{\leftarrow} , and R_L for R_R in Algorithm 1 to get decisions for packets from X_{\leftarrow} .

► **Lemma 3.** *Given the solution of the linear program, a sequence of packets X_t , and link capacity M , Algorithm 1 uses capacity $2M$ and accepts all packets fully accepted in the linear program.*

Proof. After processing the i th packet $x_i \in X_{\rightarrow}$, we denote R_L at that step $R_{L,i}$, similarly with $R_{R,i}$. We show that the channel, at time i , has capacity at least $S_{L,i}$ on the left and at least $S_{R,i}$ on the right.

■ **Algorithm 1** Algorithm accepting all fully accepted packets

Input: packet sequence X_t , capacity M , solution of LP: $S_{L,i}, S_{R,i}, y_i$.

Output: decisions to accept or reject

```

1:  $R_L = \frac{M}{2}, R_R = \frac{M}{2}$ 
2: for  $i \in [t]$  do
3:   if  $x_i \in X_{\rightarrow}$  then
4:     if  $R_L \geq x_i - y_i$  then
5:       Accept
6:        $R_L = R_L - (x_i - y_i)$ 
7:        $R_R = R_R + (x_i - y_i)$ 
8:     else
9:       Reject
10:       $R_L = R_L + y_i$ 
11:       $R_R = R_R - y_i$ 

```

When $R_{L,i}$ is large enough to accept packet x_i , we use y_i capacity from $S_{L,i}$ and $x_i - y_i$ capacity from $R_{L,i}$. The capacity y_i from the accepted packet goes to $S_{R,i+1}$ and the rest ($x_i - y_i$) of the capacity goes to $R_{R,i+1}$.

If the packet is forced to be rejected, we know that $R_{L,i} < x_i - y_i$. Since $R_{R,i} = M - R_{L,i}$, we know that $R_{R,i} > M - x_i + y_i$, and because all packets have weight smaller than M , $R_{R,i} > y_i$ follows. This means we can take y_i from $R_{R,i}$ and put add it to $S_{R,i+1}$ and remove y_i from $S_{L,i}$ (because the capacity disappeared from there) to $R_{L,i+1}$.

If the packet is fully accepted, then $x_i - y_i = 0$. That means the condition $R_{L,i} \geq x_i - y_i$ is satisfied and the algorithm accepts it. ◀

We conclude this section with two remarks.

► **Remark 4.** Lemma 3 holds for any initial distribution of $R_{L,0}$ and $R_{R,0}$ so long as $R_{L,0} + R_{R,0} = M$.

► **Remark 5.** Algorithm 1 is greedy and accepts all packets as long as $R_L \geq x_i - y_i$ (which could be suboptimal as it might not have enough capacity in R_L to accept important packets later in the sequence). However, to maintain the condition $R_L, R_R \geq 0$ in line 4 of Algorithm 1, we can substitute the conditional check $R_L \geq x_i - y_i$ with $R_R < y_i$ at any point. Then proof of Lemma 3 still holds. We note that one could use this as a heuristic to develop a better approximation as it gives some control over how greedy the algorithm is.

4 A constant approximation algorithm

Based on the insights above, we present a $(1 + \sqrt{3})$ -approximation algorithm for weighted packet selection for a link with fixed capacity M . The algorithm modifies Algorithm 1 by adding capacity to R_L and R_R , which allows us to make a wider range of decisions.

Let *little-accepted* (packets) be packets for which $\frac{y_i}{x_i} < \frac{\sqrt{3}}{1+\sqrt{3}}$ holds, and *almost-accepted* be packets for which $\frac{y_i}{x_i} \geq \frac{\sqrt{3}}{1+\sqrt{3}}$ holds.

The algorithm keeps the capacity reserve in R_L and R_R high while accepting all almost-accepted packets. It might not always have enough capacity to do so. We present procedures that deal with that situation.

In Algorithm 2, we again describe the decision making process for packets $x_i \in X_{\rightarrow}$. The decisions for packets from X_{\leftarrow} are symmetric.

■ **Algorithm 2** $(1 + \sqrt{3})$ -approximation algorithm

Input: packet sequence X_t , capacity M , solution of LP: $S_{L,i}, S_{R,i}, y_i$.

Output: decisions to accept or reject

```

1:  $R_L = M \frac{1+\sqrt{3}-1}{2}, R_R = M \frac{1+\sqrt{3}-1}{2}$ 
2: for  $i \in [t]$  do
3:   if  $x_i \in X_{\rightarrow}$  then
4:     if  $R_L - (x_i - y_i) \geq \frac{\sqrt{3}-1}{2}$  then
5:       Accept
6:        $R_L = R_L - (x_i - y_i)$ 
7:        $R_R = R_R + (x_i - y_i)$ 
8:     else if  $x_i$  is little-accepted then
9:       Reject
10:       $R_L = R_L + y_i$ 
11:       $R_R = R_R - y_i$ 
12:     else
13:       $\phi_A, \phi_R, U, R'_L, j \leftarrow \text{DIVIDE}(R_L, LP, X_t, i)$ 
14:       $U_R \leftarrow \{\}$ 
15:      if  $R'_L < 0$  then
16:         $U_R, R'_L \leftarrow \text{REJECTBIG}(X_t, U, R'_L)$ 
17:      Accept all  $x_i \in \phi_A \cup (U \setminus U_R)$ 
18:      Reject all  $x_i \in \phi_R \cup U_R$ .
19:       $R_L = R'_L$ 
20:       $R_R = (1 + \sqrt{3} - 1)M - R_L$ 
21:       $i = j$ 

```

The following lemma states that we can safely reject all little-accepted packets.

► **Lemma 6.** *Observe that all little-accepted packets can be rejected while keeping the approximation ratio below $1 + \sqrt{3}$.*

Proof. Recall that rejecting a packet x_i incurs a cost of $fx_i + m$. If x_i is little-accepted, the cost incurred by the linear program solution is $f \cdot (x_i - y_i) + m \frac{x_i - y_i}{x_i} \geq \frac{fx_i}{1 + \sqrt{3}} + \frac{m}{1 + \sqrt{3}} = \frac{1}{1 + \sqrt{3}}(fx_i + m)$. ◀

We distinguish between three phases of the algorithm. The algorithm is in the *balanced phase* if both $R_L \geq \frac{\sqrt{3}-1}{2}$ and $R_R \geq \frac{\sqrt{3}-1}{2}$. If $R_L < \frac{\sqrt{3}-1}{2}$, we say the algorithm is in the *left phase*, and if $R_R < \frac{\sqrt{3}-1}{2}$, we say the algorithm is in the *right phase*.

In the balanced phase, Algorithm 2 accepts all almost-accepted packets and those little-accepted packets that allows it to stay in the balanced phase. When the algorithm is forced to leave the balanced phase and enters the left-phase (or right-phase), it looks at future packets and tries to accept all almost-accepted packets. If this is not possible, the algorithm rejects some of them such that both of the following two conditions hold: first, the approximation ratio remains $1 + \sqrt{3}$, and second, the algorithm returns to a balanced phase. Right and left phases are handled by the functions DIVIDE (described in Algorithm 3) and REJECTBIG (described in Algorithm 4).

► **Lemma 7.** *Algorithm 2 never leaves the balanced phase after processing a little-accepted packet.*

Proof. Each little-accepted packet moves at most $\frac{1}{1+\sqrt{3}}M$ from the left side to the right side of a link, and at most $\frac{\sqrt{3}}{1+\sqrt{3}}M$ from the right side to the left side of a link.

Because $R_L + R_R = \sqrt{3}$ and any packet has a weight at most M . If $R_L - \frac{1}{1+\sqrt{3}}M < \frac{\sqrt{3}-1}{2}$, then $R_R - \frac{\sqrt{3}}{1+\sqrt{3}}M \geq \frac{\sqrt{3}-1}{2}$.

That means that rejecting a little-accepted packet from X_{\rightarrow} does not create a situation where $R_R < \frac{\sqrt{3}-1}{2}$. ◀

We have no guarantee that Algorithm 2 accepts all almost-accepted packets. Algorithms DIVIDE and REJECTBIG manage capacity when by accepting almost-accepted packets from X_{\rightarrow} leads to $R_L < \frac{\sqrt{3}-1}{2}$.

First, DIVIDE creates three sets from some future packets: ϕ_A, ϕ_R, U . Set ϕ_A contains all packets from X_{\leftarrow} , these will be accepted. Set ϕ_R contains little-accepted packets from X_{\rightarrow} , these will be rejected. Set U contains almost-accepted packets, some of them will be accepted and some rejected in a way to maintain the approximation ratio.

DIVIDE creates the sets incrementally. It simulates accepting packets from $\phi_A \cup \phi_R$ and rejecting packets from ϕ_R until one of the following stopping conditions occurs:

- the algorithm runs out of capacity (R_L would be smaller than 0)
- the algorithm returns to a balanced phase (R_L would be bigger than $\frac{\sqrt{3}-1}{2}$).
- all packets are processed

If the first condition holds, the procedure REJECTBIG is called and the procedure creates set $U_R \subset U$. Packets in ϕ_A and $U \setminus U_R$ are accepted and packets in ϕ_R and U_R are rejected.

► **Lemma 8.** *If DIVIDE returns $R'_L \geq 0$ and j , all almost-accepted packets between i and j are accepted by Algorithm 2 and either all packets are processed or $R_{R,j} \geq \frac{\sqrt{3}-1}{2}$ and $R_{L,j} \geq \frac{\sqrt{3}-1}{2}$.*

Proof. There are two reasons why DIVIDE returned $R'_L \geq 0$, either $R'_L \geq \frac{\sqrt{3}-1}{2}$ or $j = t$.

In both cases DIVIDE simulated accepting all packets from ϕ_A and U and rejecting all packets from ϕ_R , and at no time R'_L went below 0. That means that Algorithm 2 just repeats decisions of DIVIDE. ◀

Note also that $R_L + R_R = \sqrt{3}$ and all packets are smaller than M . That means that Algorithm 2 after emerging from left-phase cannot plunge to a right-phase right away.

To prove that REJECTBIG maintains the approximation ratio, we compute the cost incurred by the algorithm on packets from U_R and compare it to the cost of OPT on U .

► **Lemma 9.** *In Algorithm 2, for sets U and U_R on Line 17 holds*

$$(1 + \sqrt{3}) \sum_{x_i \in U} f \cdot (x_i - y_i) + m \frac{x_i - y_i}{x_i} \geq \sum_{x_i \in U_R} f x_i + m$$

Proof. If $R'_L \geq 0$, we know that all almost-accepted packets are accepted from Lemma 8. For $R'_L < 0$ we prove that $(1 + \sqrt{3}) \sum_{x_i \in U} (x_i - y_i) \geq \sum_{x_i \in U_R} x_i$, then we argue that the whole theorem holds.

Let $D = R_{L,i-1} - R'_L$ where R'_L is the value returned by DIVIDE in Algorithm 2 on Line 13. We know that $D \geq \frac{\sqrt{3}-1}{2}M$.

By following the changes of R'_L in DIVIDE, we get

$$\sum_{x_i \in U} x_i - y_i = D + \sum_{x_i \in \phi_R} y_i + \sum_{x_i \in \phi_A} x_i - y_i$$

■ **Algorithm 3** Function DIVIDE to create sets ϕ_A, ϕ_R , and U .

Input: packet sequence X_t , solution of LP : $S_{L,i}, S_{R,i}, y_i$, value R_L , capacity M ,

Output: sets ϕ_A, ϕ_R, U , resulting R_L

```

1:  $R_L = R_L - (x_i - y_i)$ 
2:  $\phi_A, \phi_R, U \leftarrow \{\}, \{\}, \{x_i\}$ 
3:  $j = i$ 
4: while  $R_L \geq 0$  and  $R_L < \frac{\sqrt{3}-1}{2}$  and  $j < t$  do
5:    $j = j + 1$ 
6:   if  $x_j \in X_{\rightarrow}$  and  $x_j$  is almost-accepted then
7:      $R_L = R_L - (x_j - y_j)$ 
8:      $U \leftarrow U \cup x_j$ 
9:   else if  $x_j \in X_{\rightarrow}$  and  $x_j$  is little-accepted then
10:     $R_L = R_L + y_j$ 
11:     $\phi_R \leftarrow \phi_R \cup x_j$ 
12:   else
13:     $R_L = R_L + (x_j - y_j)$ 
14:     $\phi_A \leftarrow \phi_A \cup x_j$ 
15: return  $\phi_A, \phi_R, U, R_L, j$ 

```

■ **Algorithm 4** Function REJECTBIG(t) to prune out packets from U .

Input: packet sequence X_t , set U , value R'_L

Output: set U_R , value R'_L

```

1:  $U_R \leftarrow \{\}$ 
2: while  $R'_L < \frac{\sqrt{3}-1}{2}$  do
3:    $x_k \leftarrow$  biggest packet from  $U$ 
4:    $U \leftarrow U \setminus x_k$ 
5:    $U_R \leftarrow U_R \cup \{x_k\}$ 
6:    $R'_L = R'_L + x_k$ 
7: return  $U_R, R'_L$ 

```

By this we know that $\sum_{x_i \in U} x_i - y_i \geq D$.

Algorithm REJECTBIG removes packets from U until $\sum_{x_i \in U_R} x_i \geq D$. If the condition is satisfied, we know that REJECTBIG returns U_R , because $R'_L \geq \frac{\sqrt{3}-1}{2}$.

If $|U_R| = 1$, we know that $\sum_{x_i \in U_R} x_i \leq M$, because every $x_i \leq M$. So in that case $\sum_{x_i \in U_R} x_i \leq M \leq (1 + \sqrt{3}) \frac{\sqrt{3}-1}{2} M \leq (1 + \sqrt{3}) D$.

If $|U \setminus U_R| > 1$, we know that rejecting just one packet is not enough. This means the biggest packet has weight at most D , so $\sum_{x_i \in U_R} x_i \leq 2D \leq (1 + \sqrt{3}) D$.

Now, we know that Algorithm 2 rejects less weight than the linear program times $(1 + \sqrt{3})$. It implies that $(1 + \sqrt{3}) \sum_{x_i \in U} f \cdot (x_i - y_i) \geq \sum_{x_i \in U_R} f x_i$ and leaves us to prove $(1 + \sqrt{3}) \sum_{x_i \in U} \frac{x_i - y_i}{x_i} \geq \sum_{x_i \in U_R} m$.

But we know that the packets are moved to U_R from the biggest. For every $x_k \in U_R$ and $x_l \in U$ holds $\frac{x_l - y_l}{x_l} \geq \frac{x_l - y_l}{x_k}$. That means rejecting smaller packets incurs on average bigger cost than rejecting bigger packets, so $(1 + \sqrt{3}) \sum_{x_i \in U} \frac{x_i - y_i}{x_i} \geq \sum_{x_i \in U_R} m$. ◀

We now have all the necessary ingredients to state and prove our main theorem.

► **Theorem 10.** *Weighted packet selection for a link can be approximated with a ratio $(1 + \varepsilon)(1 + \sqrt{3})$ in time $\mathcal{O}(n^\omega \cdot \frac{1}{\varepsilon} \cdot \log \frac{M_{\max}}{x_{\min}})$, where ω is the exponent of n in matrix multiplication.*

Proof. We estimate the capacity according to Lemma 1 and for every estimate we solve the linear program (1) and run Algorithm 2. The solution is the output of Algorithm 2 with the smallest cost.

We know that $x_{\min}(1 + \varepsilon)^{\frac{1}{\varepsilon} \cdot \log \frac{M_{\max}}{x_{\min}}} \geq M_{\max}$. That means we solve the linear program and run Algorithm 2 at most $\frac{1}{\varepsilon} \cdot \log \frac{M_{\max}}{x_{\min}}$ times.

From Lemma 2 we know that the solution of the linear program with parameter M is a lower bound for the OPT_M^R .

From Lemma 3, we know that Algorithm 2 accepts all fully-accepted packets. The algorithm can reject any little-accepted packets by Lemma 6. In balanced phase it accepts all almost-accepted packets and never leaves the phase after seeing little-accepted packets (Lemma 7). Lemma 9 says even in a left (or right) phase the approximation ratio on almost-accepted packets is $1 + \sqrt{3}$. This means Algorithm 2 is $(1 + \sqrt{3})$ -approximation algorithm for the solution of the linear program. Moreover, the algorithm uses $1 + \sqrt{3}$ times more capacity than the linear program.

Using Lemma 1, we find that the selected solution is a $(1 + \varepsilon)(1 + \sqrt{3})$ -approximation of the weighted packet selection for a link problem. ◀

5 Hardness

In this section, we show that weighted packet selection for a link is generally NP-hard.

► **Theorem 11.** *Weighted packet selection for a link is NP-hard.*

Proof. We show a reduction from the subset sum problem, which is known to be NP-hard [2]. In the subset sum problem, we are given a multiset of integers $\mathcal{I} := \{i_1, i_2, \dots, i_n\}$ and a target integer S . The goal is to find a subset of \mathcal{I} with a sum of S .

Consider the following question in the weighted packet selection for a link problem: "is the cost below a given value?" We show this question is NP-hard.

We set the constants to $m = 0$ and $f = \frac{3}{4}$. We create a packet sequence consisting of i_1, i_2, \dots, i_n where the j th packet in the sequence has weight i_j which is the j th element in \mathcal{I} . These packets all go from left to right. Then we add a packet of weight S going from right to left.

Suppose that there exists $\mathcal{I}' \subseteq \mathcal{I}$, such that $\sum_{j \in \mathcal{I}'} i_j = S$. Then we show that the cost is at most $\frac{1}{4}S + \frac{3}{4} \sum_{j \in \mathcal{I}} i_j$.

The solution reaching that cost is as follows: players start with capacity S on the right and accept all packets from \mathcal{I}' and then accept the last packet of weight S . The cost is then $S + \frac{3}{4} \sum_{j \in \mathcal{I} \setminus \mathcal{I}'} i_j$. Since $\sum_{j \in \mathcal{I}'} i_j = S$, the bound holds.

Now, suppose that there is no subset of \mathcal{I} summing to S . Let $\mathcal{A} \subseteq \mathcal{I}$ be any set with sum A . The cost for the packets going from left to right is $A + \frac{3}{4} \sum_{j \in \mathcal{I} \setminus \mathcal{A}} i_j = \frac{1}{4}A + \frac{3}{4} \sum_{j \in \mathcal{I}} i_j$. Depending whether the last packet was accepted or rejected; or $A < S$ or $A > S$, we need to

add $\min(\max(S - A, 0), \frac{3}{4}S)$. Since $A \neq S$, we know that

$$\begin{aligned} \frac{1}{4}A + \frac{3}{4} \sum_{j \in \mathcal{I}} i_j + \min(\max(S - A, 0), \frac{3}{4}S) &> \frac{1}{4}S + \frac{3}{4} \sum_{j \in \mathcal{I}} i_j \\ \min(\max(S - A, 0), \frac{3}{4}S) &> \frac{1}{4}(S - A) \end{aligned}$$

which means that weighted packet selection for a link is NP-hard. \blacktriangleleft

6 Extensions

We highlight two natural and interesting directions to generalise our approach from a link to a network.

6.1 Cyclic redistribution of capacity to reduce cost

Suppose player u on link (u, v) is incident to ≥ 2 links (let us call one of the incident links (u, w)). From our definition of rechargeable links (see Section 1), we know it is not possible for u to increase the capacity on the (u, v) link by transferring excess capacity from (u, w) . However, if (u, v) and (u, w) are part of a larger cycle in the network, u can send excess capacity from link to link in a cyclic fashion starting from (u, w) link and ending at (u, v) while maintaining the invariant that the total capacity on each link as well as the sum of all the capacities of a player on their incident links remains the same. This can be done at any point in time without the need to transfer packets. We call this cyclic redistribution (note this is possible in payment channel networks [11, 10, 4]) and illustrate it with an example in Figure 2. In some situations, especially if the cost of destroying and recreating a link is extremely large, the possibility of cheaply shifting capacities in cycles can reduce the overall cost of the algorithm.

Let us denote the cost of decreasing capacities by x on the right and increasing it by x on the left using cyclic redistribution by $C(fx + m)$ for some $C \geq 1$ (one can view C as a function of the length of the cycle one sends the capacities along).

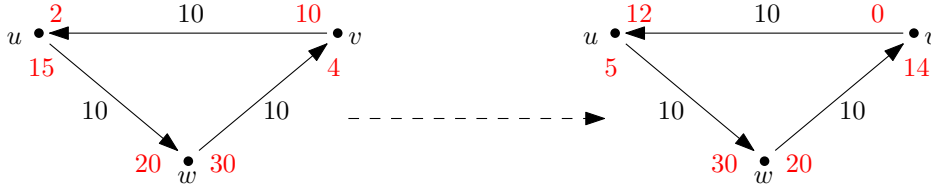
Here, we sketch an approximation algorithm that solves the weighted packet selection for a link problem with the possibility of cyclic redistribution. Note that our sketch is not precise, we simply modify Algorithm 2 where constants are already optimised for the basic problem.

We modify the linear program by adding variables $o_i, i \in [t]$ with constraints $0 \leq o_i \leq M$. The variable o_i denotes the capacity that was shifted from one side to the other before the algorithm processes packet x_i . We also modify the capacity constraints in the following way (for the case where $x_{i-1} \in X_{\leftarrow}$ and $x_i \in X_{\leftarrow}$): $S_{L,i} = S_{L,i-1} - o_i + y_{i-1}$ and $S_{R,i} = S_{R,i-1} + o_i - y_{i-1}$. We change the signs of variables for the other cases. Finally, we add $\sum_i C(f o_i + \frac{o_i}{M} m)$ to the objective in Equation (1).

We divide the algorithm into epochs. We sum all o_i in the current epoch. If the sum is above $\frac{1}{1+\sqrt{3}}M$ we perform cyclic distribution if needed and start a new epoch. Note that in the current epoch, OPT already paid at least $Cf \frac{M+m}{1+\sqrt{3}}$, so, we can move M capacity, incurring a cost at most $1 + \sqrt{3}$ times bigger than OPT for cyclic redistribution.

To deal with capacity changes inside each epoch, we increase R_L and R_R . We initialise them in a way that they absorb changes of capacity in the first epoch of our algorithm. After an epoch, we reset them by cyclic redistribution such that they absorb changes of capacity

in the next. The increase in R_L and R_R is at most $\frac{1}{1+\sqrt{3}}M$. These changes increase the approximation ratio of our algorithm from $1 + \sqrt{3}$ to $1 + \sqrt{3} + \frac{1}{1+\sqrt{3}}$, which is $\frac{1+3\sqrt{3}}{2}$.



■ **Figure 2** The graph on the left depicts three players u, v, w connected in a cycle. The red numbers by each link represent the capacity of a player in a certain link. Player u can increase their capacity by 10 on the (u, v) link by first sending the excess capacity of 10 to w along the (u, w) link. Then player w sends the excess capacity of 10 to v along (w, v) . Finally, player v sends capacity of 10 back to u along (v, u) . The graph on the right depicts the updated capacities of each player on each link after cyclic redistribution.

6.2 Graphs with a few long packet requests

We can extend weighted packet selection from a single link to a network. We first modify the definition of a packet such that, in addition to having a non-negative weight, it now has a sequence of (directed) links it needs to be processed by. These links form the path that packet takes through the network. The packet has to be accepted or rejected by *all* the links in its routing path. We call a packet *long* if it passes through more than one link.

Thus far, we showed an approximation algorithm that solves the problem if all packets only go through 1 link. Suppose we are given the situation where we can bound the number of long packets, say by ℓ . Given a network and packet sequence for which we know only ℓ are long, we can approximate the extended problem with approximation ratio $(1 + \varepsilon)(1 + \sqrt{3})$ in time 2^ℓ times the time needed for one link by simply trying all to accept all subsets of paths of long packets. If the packet is accepted or rejected, we can reflect it in the linear program by requiring $y_i = x_i$. Then Algorithm 2 surely accepts this packet and the condition that the packet needs to be accepted by all links it passes through is satisfied.

7 Discussion and future work

We initiated the study of weighted packet selection over a rechargeable capacitated link, a natural algorithmic problem e.g., describing the routing of financial transactions in cryptocurrency networks. We showed that this problem is NP-hard and provided a constant factor approximation algorithm.

We understand our work as a first step, and believe that it opens several interesting avenues for future research. In particular, it remains to find a matching lower bound for the achievable approximation ratio, and to study the performance of our algorithm in practice. More generally, it would be interesting to study the online version of the weighted packet selection problem, and explore competitive algorithms. This version of the problem, when extended to a network, can be seen as a novel version of the classic online call admission problem [3].

Acknowledgments. We thank Mahsa Bastankhah and Mohammad Ali Maddah-Ali for fruitful discussions about different variants of the problem.

References

- 1 Raiden network. <https://raiden.network/>, 2017.
- 2 Sanjeev Arora and Boaz Barak. *Computational Complexity: A Modern Approach*. Cambridge University Press, USA, 1st edition, 2009.
- 3 James Aspnes, Yossi Azar, Amos Fiat, Serge Plotkin, and Orli Waarts. On-line routing of virtual circuits with applications to load balancing and machine scheduling. *Journal of the ACM (JACM)*, 44(3):486–504, 1997.
- 4 Zeta Avarikioti, Krzysztof Pietrzak, Iosif Salem, Stefan Schmid, Samarth Tiwari, and Michelle Yeo. Hide and seek: Privacy-preserving rebalancing on payment channel networks. In *Proc. Financial Cryptography and Data Security (FC)*, 2022.
- 5 Chandra Chekuri, Sanjeev Khanna, and F Bruce Shepherd. The all-or-nothing multicommodity flow problem. In *Proceedings of the 36th Annual ACM Symposium on Theory of Computing (STOC)*, pages 156–165, 2004.
- 6 Michael B. Cohen, Yin Tat Lee, and Zhao Song. Solving linear programs in the current matrix multiplication time. *J. ACM*, 68(1):3:1–3:39, 2021.
- 7 Christian Decker and Roger Wattenhofer. A fast and scalable payment network with bitcoin duplex micropayment channels. In *Symposium on Self-Stabilizing Systems (SSS)*, pages 3–18. Springer, 2015.
- 8 Maya Dotan, Yvonne-Anne Pignolet, Stefan Schmid, Saar Tochner, and Aviv Zohar. Survey on blockchain networking: Context, state-of-the-art, challenges. In *Proc. ACM Computing Surveys (CSUR)*, 2021.
- 9 Piyush Kumar Gupta and Panganamala Ramana Kumar. The capacity of wireless networks. *IEEE Trans. Inf. Theory*, 46:388–404, 2000.
- 10 Rami Khalil and Arthur Gervais. Revive: Rebalancing off-blockchain payment networks. In Bhavani M. Thuraisingham, David Evans, Tal Malkin, and Dongyan Xu, editors, *Proceedings of the 2017 ACM SIGSAC Conference on Computer and Communications Security, CCS 2017, Dallas, TX, USA, October 30 - November 03, 2017*, pages 439–453. ACM, 2017.
- 11 Rene Pickhardt and Mariusz Nowostawski. Imbalance measure and proactive channel rebalancing algorithm for the lightning network. In *IEEE International Conference on Blockchain and Cryptocurrency, ICBC 2020, Toronto, ON, Canada, May 2-6, 2020*, pages 1–5. IEEE, 2020.
- 12 Joseph Poon and Thaddeus Dryja. The bitcoin lightning network: Scalable off-chain instant payments. <https://lightning.network/lightning-network-paper.pdf>, 2015.
- 13 Prabhakar Raghavan and Clark D Thompson. Provably good routing in graphs: regular arrays. In *Proceedings of the seventeenth annual ACM symposium on Theory of computing*, pages 79–87, 1985.