

Self-Adjusting Linear Networks with Ladder Demand Graph

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Abstract. Self-adjusting networks (SANs) have the ability to adapt to communication demand by dynamically adjusting the workload (or demand) embedding, i.e., the mapping of communication requests into the network topology. SANs can thus reduce routing costs for frequently communicating node pairs by paying a cost for adjusting the embedding. This is particularly beneficial when the demand has structure, which the network can adapt to. Demand can be represented in the form of a demand graph, which is defined by the set of network nodes (vertices) and the set of pairwise communication requests (edges). Thus, adapting to the demand can be interpreted by embedding the demand graph to the network topology. This can be challenging both when the demand graph is known in advance (offline) and when it revealed edge-by-edge (online). The difficulty also depends on whether we aim at constructing a static topology or a dynamic (self-adjusting) one that improves the embedding as more parts of the demand graph are revealed. Yet very little is known about these self-adjusting embeddings.

In this paper, the network topology is restricted to a line and the demand graph to a ladder graph, i.e., a $2 \times n$ grid, including all possible subgraphs of the ladder. We present an online self-adjusting network that matches the known lower bound asymptotically and is 12-competitive in terms of request cost. As a warm up result, we present an asymptotically optimal algorithm for the cycle demand graph. We also present an oracle-based algorithm for an arbitrary demand graph that has a constant overhead.

Keywords: Ladder graph · Self-adjusting networks · Traffic patterns · online algorithms.

1 Introduction

Traditional networks are static and demand-oblivious, i.e., designed without considering the communication demand. While this might be beneficial for all-to-all traffic, it doesn't take into account temporal or spatial locality features in demand. That is, sets of nodes that temporarily cover the majority of communication requests may be placed diameter-away from each other in the network topology. This is a relevant concern as studies on datacenter network traces have shown that communication demand is indeed bursty and skewed [3].

Self-adjusting networks (SANs) are optimized towards the traffic they serve. SANs can be static or dynamic, depending on whether it is possible to reconfigure

the embedding (mapping of communication requests to the network topology) in between requests, and offline or online, depending on whether the sequence of communication requests is known in advance or revealed piece-wise. In the online case, we assume that the embedding can be adjusted in between requests at a cost linear to the added and deleted edges, thus, bringing closer frequently communicating nodes. Online algorithms for SANs aim to reduce the sum of routing and reconfiguration (re-embedding) costs for any communication sequence.

We can express traffic in the form of a demand graph that is defined by the set of nodes in the network and the set of pairwise communication requests (edge set) among them. Knowing the structure of the demand graph could allow us to further optimize online SANs, even though the demand is still revealed online. That is, by re-embedding the demand graph to the network we optimize the use of network resources according to recent patterns in demand.

To the best of our knowledge, the only work on demand graph re-embeddings to date is [2], where the network topology is a line and the demand graph is also a line. The authors presented an algorithm that serves $m = \Omega(n^2)$ requests at cost $O(n^2 \log n + m)$ and showed that this complexity is the lower bound. The problem is inspired by the Itinerant List Update Problem [11] (ILU). To be more precise, the problem in [2] appears to be the restricted version of the online Dynamic Minimum Linear Arrangement problem, which is another reformulation of ILU.

Contributions. In this work, we take the next step towards optimizing online SANs for more general demand graphs. We restrict the network topology to a line, but assume that the demand graph is a ladder, i.e., a $2 \times n$ grid. We assume that before performing a request, we can re-adjust the line graph by performing several swaps of two neighbouring nodes, paying one for each swap. We present a 12-competitive online algorithm that embeds a ladder demand graph to the line topology, thus, asymptotically matching the lower bound in [2]. This algorithm can be applied to any demand graph that is a subgraph of the ladder graph and that when all edges of the demand graph are revealed the topology is optimal and no more adjustments occur. We also optimally solve the case of cycle demand graphs, which is a simple generalization of the line demand graph, but is not a subcase of the ladder due to odd cycles. Finally, we provide a generic algorithm for arbitrary demand graphs, given an oracle that computes an embedding with the cost of requests bounded by the bandwidth.

A solution for the ladder is the first step towards the $n \times m$ grid demand graph. Moreover, a ladder (and a cycle) has a constant *bandwidth*, i.e., a minimum value over all embeddings onto a target line graph of a maximal path between the ends of an edge (request). It can be shown that given a demand graph G the best possible complexity per request is the bandwidth.

Related work. Avin et al. [2], consider a fixed line (host) network and a line demand graph. Their online algorithm re-embeds the demand graph to the host line graph with minimum number of swaps on the embedding. Both [1,6] present constant-competitive online algorithms for a fixed and complete binary tree, where nodes can swap and the demand is originating only from the source. However, these two works do not consider a specific demand graph. Moreover,

[5] studied optimal but static and bounded-degree network topologies, when the demand is known. Self-adjusting networks have been formally organized and surveyed in [7]. Other existing online SAN algorithms consider different models. The most distinct difference is our focus on online re-embedding while keeping a fixed host graph (i.e., a line) compared to other works that focus on changing the network topology. The latter is what, for example, SplayNet [12] is proposing, where tree rotations change the form of the binary search tree network, without optimizing for a specific family of demand patterns.

Online demand graph re-embedding also relates to dynamically re-allocating network resources to follow traffic patterns. In [4], the authors consider a fixed set of clusters of bounded size, which contain all nodes and migrate nodes online according to the communication demand. But more broadly, [8] assumes a fixed grid network and migrates tasks according to their communication patterns.

Also, relevant problems, from a migration point of view, are the classic list update problem (LU) [13], the related Itinerant List Update (ILU) problem [11], and the Minimum Linear Arrangement (MLA) problem [10]. In contrast to those problems, we study an online problem where requests occur between nodes.

Roadmap. Section 2 describes the model and background. Section 3 contains the summary of our three contributions (ladder, cycle, general demand graph) and their high-level proofs. Section 4 presents the algorithm and analysis for ladder demand graphs. Some technical details are deferred to the appendix.

2 Model and Background

Let us introduce the notation that we are going to use throughout the paper. Let $V(H)$ and $E(H)$ be the sets of vertices and edges in graph H , respectively. Sometimes, we just use V and E if the graph H is obvious from the context. Let $d_H(u, v)$ be the distance between u and v in graph H .

Let N be the network topology and σ be a sequence of pairwise communication requests between nodes in N . Let the demand graph G be the graph built over the nodes in N and the pairs of nodes that appear in σ , i.e. $G = (V(N), \{\sigma_i = (s_i, d_i) \mid \sigma_i \in \sigma\})$. We assume that the demand graph is of a certain type and our overall goal will be to embed the demand graph G onto the actual network topology N at a minimum cost. This is non-trivial as requests are selected from G by an online adversary and G is not known in advance. In the following, we formalize demand graph embedding and topology reconfiguration.

A configuration (or an embedding) of G (the demand graph) in a graph N (the host network) is an injection of $V(G)$ into $V(N)$; $C_{G \rightarrow N}$ denotes the set of all such configurations. A configuration $c \in C_{G \rightarrow N}$ is said to serve a communication request $(u, v) \in E(G)$ at the cost $d_N(c(u), c(v))$. A finite communication sequence $\sigma = (\sigma_1, \dots, \sigma_m)$ is served by a sequence of configurations $c_0, c_1, \dots, c_m \in C_{G \rightarrow N}$. The cost of serving σ is the sum of serving each σ_i in c_i plus the reconfiguration cost between subsequent configurations c_i and c_{i+1} . The reconfiguration cost between c_i and c_{i+1} is the number of migrations necessary to change from c_i to c_{i+1} ; a migration swaps the images of two neighbouring nodes u and v under c in N . Moreover, $E_i = \{\sigma_1, \dots, \sigma_i\}$ denotes the first i requests of σ interpreted

as a set of edges on V . We present algorithms for an online self-adjusting linear network: a network whose topology forms a 1-dimensional grid, i.e., a line.

Definition 1 (Working Model). Let G be the demand graph, n be the number of vertices in G , $N = (\{1, \dots, n\}, \{(1, 2), (2, 3), \dots, (n-1, n)\})$ be a line (or list) graph L_n (host network), c be a configuration from $C_{G \rightarrow N}$, and σ be a sequence of communication requests. The cost of serving $\sigma_i = (u, v) \in \sigma$ is given by $|c(u) - c(v)|$, i.e., the distance between u and v in N . Migrations can occur before serving a request and can only occur between nodes configured on adjacent vertices in N .

In the following we introduce notions relevant to our new results.

Definition 2. A correct embedding of a graph G into graph N is an injective mapping $\varphi : V(G) \rightarrow V(N)$ that preserves edges, i.e.

$$\begin{cases} \forall u, v \in V(G) \text{ with } u \neq v \Rightarrow \varphi(u) \neq \varphi(v) \\ (u, v) \in E(G) \Rightarrow (\varphi(u), \varphi(v)) \in E(N) \end{cases}$$

Definition 3 (Bandwidth). Given a graph G , the Bandwidth of an embedding $c \in C_{G \rightarrow L_n}$ is equal to the maximum over all edges $(u, v) \in E$ of $|c(u) - c(v)|$, i.e., the distance between u and v on L_n . $\text{Bandwidth}(G)$ is the minimum bandwidth over all embeddings from $C_{G \rightarrow L_n}$.

Remark 1 The Bandwidth computation of an arbitrary graph is an NP-hard problem [9].

To save the space, we typically omit the proofs of lemmas and theorems in this paper and put them in Appendix C. Here we define the $2 \times n$ grid or ladder graph for which we get the main results of our paper.

Definition 4. A graph $\text{Ladder}_n = (V, E)$ is represented as follows. The vertices V are the nodes of the grid $2 \times n = \{(1, 1), (1, 2), \dots, (1, n), (2, 1), (2, 2), \dots, (2, n)\}$. There is an edge between vertices (x_1, y_1) and (x_2, y_2) iff $|x_1 - x_2| + |y_1 - y_2| = 1$.

Lemma 1. $\text{Bandwidth}(\text{Ladder}_n) = 2$.

Proof. The bandwidth is greater than 1, because there are nodes of degree three. The bandwidth of 2 can be achieved via the “level-by-level” enumeration as shown on the figure.

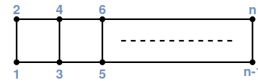


Fig. 1: Optimal ladder numeration.

Lemma 2. For each subgraph S of a graph G , $\text{Bandwidth}(S) \leq \text{Bandwidth}(G)$.

2.1 Background

Let us overview the previous results from [2]. In that work, both the demand and the host graph (network topology) were the line graph L_n on n vertices. It was shown that there exists an algorithm that performs $O(n^2 \log n)$ migrations in total, while serving the requests themselves in $O(1)$. By that, if the number of requests is $\Omega(n^2 \log n)$ then each request has $O(1)$ amortized cost.

Theorem 1 (Avin et al. [2]). *Consider a linear network L_n and a linear demand graph. There is an algorithm such that the total time spent on migrations is $O(n^2 \log n)$, while each request is performed in $O(1)$ omitting the migrations.*

We give an overview of this algorithm. At each moment in time, we know some subgraph of the line demand graph. For each new communication request, there are two cases: 1) the edge from the demand graph is already known — then, we do nothing; 2) the new edge is revealed. In the second case, this edge connects two connected components. We just move the smallest component on the line network closer to the largest component. The move of each node in one reconfiguration does not exceed n . Since, the total number of reconfigurations in which the node participates does not exceed $\log n$, we have $O(n^2 \log n)$ upper bound on the algorithm. From [2], $\Omega(n^2 \log n)$ is also the lower bound on the total cost. Thus, the algorithm is asymptotically optimal in the terms of complexity.

Corollary 1. *If $|\sigma| = \Omega(n^2 \log n)$ the amortized service cost per request is $O(1)$.*

The algorithms are not obliged to perform migrations at all, but the sum of costs for $\Theta(n^2)$ requests can be lower-bounded with $\Omega(n^2 \log n)$.

Theorem 2 (Lower bound, Avin et al. [2]). *For every online algorithm ON there is a sequence of requests σ_{ON} of length $\Theta(n^2)$ with the demand graph being a line, such that $\text{cost}(ON(\sigma_{ON})) = \Omega(n^2 \log n)$.*

That implies $\Omega(\log n)$ optimality factor since any offline algorithm knowing the whole request sequence σ in advance can simply reconfigure the network to match the (line) demand graph by paying $\Theta(n^2)$ in the worst case.

3 Summary of contributions

In this work we present self-adjusting networks with a line topology for a demand graph that is either a cycle, or a $2 \times n$ grid (ladder), or an arbitrary graph. We study offline and online algorithms on how to best embed the demand graph on the line graph, such that the embedding cost is minimized. The online case is more challenging, as the demand graph is revealed edge-by-edge and the embedding changes, with a cost. The result for the cycle follows from [2] almost directly. However the result for the ladder is non-trivial and requires new techniques; it is not simple to reconfigure a subgraph on a $2 \times n$ grid after revealing a new edge in order to get $O(n^2 \log n)$ cost of modifications in total. We give an overview of each case below.

3.1 Cycle demand graph

We start with the following observation. Let C_n be a cycle graph on n vertices, i.e., $E(C_n) = \{(1, 2), \dots, (n-1, n), (n, 1)\}$. Then, $\text{Bandwidth}(C_n) = 2$. We give a brief description of how the algorithm works. We start with the same algorithm as for the line (Section 2.1): while the number of revealed edges is not more than $n-1$, we can emulate the algorithm for the line. When the last edge appears we restructure the whole embedding in order to get bandwidth 2, which is the cycle bandwidth. For the whole restructuring using swaps, we pay no more than $O(n^2)$. This cost is less than the total time spent on the reconstruction $\Omega(n^2 \log n)$.

Theorem 3. *Suppose the demand graph is C_n . There is an algorithm such that the total cost spent on the migrations is $O(n^2 \log n)$ and each request is performed in $O(1)$. In particular, if the number of requests is $\Omega(n^2 \log n)$ each request has $O(1)$ amortized cost.*

The full proof appears in Appendix A.

Remark 2 *The lower bound with $\Omega(n^2 \log n)$ that was presented for a line demand graph still holds in the case of a cycle, since the cycle contains the line as the subgraph. Thus, our algorithm is optimal.*

3.2 Ladder demand graph

Now, we state the main result of the paper — the algorithm for the case when the demand graph is an ladder.

Theorem 4. *Suppose a demand graph is an ladder. There is an algorithm such that the total cost spent on the migrations is $O(n^2 \log n)$ and each request is performed in $O(1)$. In particular, if the number of requests is $\Omega(n^2 \log n)$ each request has $O(1)$ amortized cost.*

We provide a brief description of the algorithm. We say that an ladder has n levels from left to right: i.e., the nodes $(1, x)$ and $(2, x)$ are on the same level x (see Figure 1). On a high-level, we use the same algorithmic approach as in Theorem 1 for the line demand graph. The main difference is that instead of embedding the demand graph right away onto the line network, at first, we “quasi-embed” the graph onto the $2n$ -ladder graph, which then we embed onto the line. By “quasi-embedding” we mean a relaxation of the embedding defined earlier: at most **three** vertices of the demand graph are mapped on each level of the ladder.

Suppose for a moment that we have a dynamic algorithm that quasi-embeds the graph onto the $2n$ -ladder. Given this quasi-embedding we can then embed the $2n$ -ladder onto the line L_n . We consequently go through from level 1 to level $2n$ of our ladder and map (at most three) vertices from the level to the line in some order (see Theorem 1). Such a transformation from the ladder to the line costs only a constant multiplication overhead.

We explain briefly how to design a dynamic quasi-embedding algorithm with the desired complexity. At first, we present a static quasi-embedding algorithm, i.e., we are given a subgraph of the ladder and we need to quasi-embed it. This algorithm consists of three parts: embed a tree, embed a cycle, embed everything together. To embed a tree we find a special path in it, named trunk. We embed this trunk from left to right: one vertex per level. All the subgraphs connected to trunk are pretty simple and can be easily quasi-embedded in parallel to the trunk (see Figure 2). To embed a cycle we just have to decide which orientation it should have. To simplify the algorithm we embed only the cycles of length at least 6, omitting the cycles of length 4. This decision increases the multiplicative constant of the cost. Finally, we embed the whole graph: we construct its cycle-tree decomposition and embed cycles and trees one by one from left to right.

Now, we give a high-level description of our dynamic algorithm. We maintain the invariant that all the components are quasi-embedded. When an already

served request appears, we do nothing. The complication comes from a newly revealed edge-request. There are two cases. The first one is when the edge connects nodes in the same component — thus, there is a cycle. We redo only the part of the quasi-embedding of the component around the new cycle; the rest of the component remains. In the second case, the edge connects two components. We move the smaller component to the bigger one as in Theorem 1. The bigger component does not move and we redo the quasi-embedding of the smaller one.

Now, we briefly calculate the complexity of the dynamic algorithm. For the requests of the first case, if the nodes are on the cycle for the first time (this event happens



Fig. 2: Quasi-correct embedding of a tree

only once for each node), we pay $O(n)$ for it. Otherwise, there are already nodes in the cycle. In this case we make sure to re-embed the existing cycle in a way that all the nodes are moved for a $O(1)$ distance. As for the neighboring nodes, it can be shown that each node is moved only once as a part of the cycle neighborhood, so we also bound this movement with $O(n)$ cost. This gives us $O(n^2)$ complexity in total — each node is moved by at most $O(n)$. For the requests of the second case, we always move the smaller component and, thus, we pay $O(n^2 \log n)$ in total: each node can be moved by $O(n)$ at most $O(\log n)$ times, i.e., any node can be at most $\log n$ times in the “smaller” component. Our algorithm matches the lower bound, since the ladder contains L_n as a subgraph.

3.3 General graph

We finish the list of contributions with a general result; the case where the demand graph is an arbitrary graph G . The full proofs are available at Appendix D.

Theorem 5. *Suppose we are given a (demand) graph G and an algorithm B , that for any subgraph S of G outputs an embedding $c \in C_{S \rightarrow L_{|V(G)|}}$ with bandwidth less than or equal to $\lambda \cdot \text{Bandwidth}(G)$ for some λ . Then, for any sequence of requests σ with a demand graph G there is an algorithm that serves σ with a total cost of $O(|E(G)| \cdot |V(G)|^2 + \lambda \cdot \text{Bandwidth}(G) \cdot |\sigma|)$. In particular, if the number of requests is $\Omega(|E(G)| \cdot |V(G)|^2)$ each request has $O(\lambda \cdot \text{Bandwidth}(G))$ amortized cost.*

Here we give a brief description of the algorithm. Suppose that the current configuration c_i is the embedding of the current demand graph G_i onto $L_{|V(G)|}$ after i requests. Now, we need to serve a new request in $\lambda \cdot \text{Bandwidth}(G_i) \leq \lambda \cdot \text{Bandwidth}(G)$. If the corresponding edge already exists in the demand graph, we simply serve the request without the reconfiguration. Now, suppose the request reveals a new edge and we get the demand graph G_{i+1} . Using the algorithm B we get the configuration (embedding) c_{i+1} that has $\lambda \cdot \text{Bandwidth}(G_{i+1}) \leq \lambda \cdot \text{Bandwidth}(G)$. To serve the request fast, we should rebuild the configuration c_i into the configuration c_{i+1} . By using the swap operations on the line we can get from c_i to c_{i+1} in $O(|V(G)|^2)$ operations: each vertex moves by at most $V(G)$. After the reconfiguration we can serve the request with the desired cost.

A new edge appears at most $|E(G)|$ times while the reconfiguration costs $|V(G)|^2$. Each request is served in $\lambda \cdot \text{Bandwidth}(G)$. Thus, the total cost of requests σ is $O(|E(G)| \cdot |V(G)|^2 + \lambda \cdot \text{Bandwidth}(G) \cdot |\sigma|)$.

Lemma 3. *Given a demand graph G . For each online algorithm ON there is a request sequence σ_{ON} such that ON serves each request from σ_{ON} for a cost of at least $\text{Bandwidth}(G)$.*

4 Embedding a ladder demand graph

We present our algorithms for embedding a demand graph that is a subgraph of the ladder graph ($2 \times n$ -grid) on the line graph. We first present the offline case, where the demand graph is known in advance (Section 4.1). Then we present the dynamic case, where requests are revealed online, revealing also the demand graph and thus possibly changing the current embedding (Section 4.2). Finally, we discuss the cost of the dynamic case in Section 4.3.

Though our final goal is to embed a demand graph into the line, we will first focus on how to embed a partially-known demand graph into $Ladder_N$, where N is large enough to make the embedding possible, i.e., not more than $2n$. When we have such an embedding one might construct an embedding from $Ladder_N$ into $Line_n$, simply composing it with a level by level (see the proof of Lemma 1) embedding of $Ladder_N$ to $Line_{2N}$ and then by omitting empty images we get $Line_n$. Such a mapping of $Ladder_N$ to $Line_{2N}$ enlarges the bandwidth for at most a factor of 2, but significantly simplifies the construction of our embedding.

Definition 5. *An ladder graph l consists of two line-graphs on n vertices l_1 and l_2 with additional edges between the lines: $\{(l_1[i], l_2[i]) \mid i \in [n]\}$, where $l_j[i]$ is the i -th node of the line-graph l_j . We call the set of two vertices, $\{l_1[i], l_2[i]\}$, the i -th level of the ladder and denote it as $level_{Ladder_n}(i)$ or just $level(i)$ if it is clear from the context. We refer to $l_1[i]$ and $l_2[i]$ as $level(i)[1]$ and $level(i)[2]$, respectively. We say that $level\langle v \rangle = i$ for $v \in V(Ladder_n)$ if $v \in level_{Ladder_n}(i)$. We refer to l_1 and l_2 as the sides of the ladder.*

4.1 Static quasi-embedding

We start with one of the basic algorithms — how to quasi-embed on $Ladder_N$ with large N any graph that can be embedded in $Ladder_n$. We present a tree and cycle embedding and then we show how to combine them in embedding a general component (by first doing a cycle-tree decomposition). The whole algorithm is presented in Appendix B.1.

Tree embedding In this case, our task is to embed a tree on a ladder graph. We start with some definitions and basic lemmas.

Definition 6. *Consider some correct embedding φ of a tree T into $Ladder_n$. Let $r = \arg \max_{v \in V(T)} level\langle \varphi(v) \rangle$ and $l = \arg \min_{v \in V(T)} level\langle \varphi(v) \rangle$ be the “rightmost” and “leftmost” nodes of the embedding, respectively. The trunk of T is a path in T connecting l and r . The trunk of a tree T for the embedding φ is denoted with $trunk_\varphi(T)$.*

Definition 7. Let T be a tree and φ be its correct embedding into $Ladder_n$. The level i of $Ladder_n$ is called occupied if there is a vertex $v \in V(T)$ on that level, i.e., $\varphi(v) \in level_{Ladder_n}(i)$.

Statement 1 For every occupied level i there is $v \in trunk_\varphi(T)$ such that $v \in level(i)$.

Proof. By the definition of the trunk, an image goes from the minimal occupied level to the maximal. It cannot skip a level since the trunk is connected and the correct embedding preserves connectivity.

The trunk of a tree in an embedding is a useful concept to define since the following hold for it. The proofs for the lemmas in this section appear in Appendix C.

Lemma 4. Let T be a tree correctly embedded into $Ladder_n$ by some embedding φ . Then, all the connected components in $T \setminus trunk_\varphi(T)$ are line-graphs.

Lemma 5. For the tree T and for each node v of degree three (except for maximum two of them) we can verify in polynomial time if for any correct embedding φ , $trunk_\varphi(T)$ passes through v or not.

Support nodes are the nodes of two types: either a node of degree three without neighbours of degree three or a node that is located on some path between two nodes with degree three. The path through passing through all support nodes is called *trunk core*. We denote this path for a tree T as $trunkCore(T)$. Intuitively, the trunk core consists of vertices that lie on a trunk of any embedding. It can be proven that the support nodes appear in the trunk of every correct embedding (proof appears in the appendix).

Definition 8. Let T be a tree. All the connected components in $T \setminus trunkCore(T)$ are called simple-graphs of tree T .

Lemma 6. The simple-graphs of a tree T are line-graphs.

Definition 9. The edge between a simple-graph and the trunk core is called a leg. The end of a leg in the simple-graph is called a head of the simple-graph. The end of a leg in the trunk core is called a foot of the simple-graph. If you remove the head of a simple-graph and it falls apart into two connected components, such simple-graph is called two-handed and those parts are called its hands. Otherwise, the graph is called one-handed, and the sole remaining component is called a hand. If there are no nodes in the simple-graph but just a head it is called zero-handed.

Definition 10. A simple-graph connected to some end node of the trunk core is called exit-graph. A simple-graph connected to an inner node of the trunk core is called inner-graph.

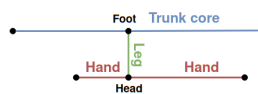


Fig. 3: Hands, Legs, and Trunk core.

Please note that the next definition is about a much larger ladder graph, $Ladder_N$, rather than $Ladder_n$. Here, N is equal to $2n$ to make sure that we have enough space to embed.

Definition 11. An embedding $\varphi : V(G) \rightarrow V(Ladder_N)$ of a graph G into $Ladder_N$ is called quasi-correct if:

- $(u, v) \in E(G) \Rightarrow (\varphi(u), \varphi(v)) \in E(Ladder_N)$, i.e., images of adjacent vertices in G are adjacent in the grid.
- There are no more than **three** nodes mapped into each level of $Ladder_N$, i.e., the two grid nodes on each level are the images of no more than three nodes.

We can think of a quasi-correct embedding as an embedding into levels of the grid with no more than three nodes embedded to the same level. Then, we can compose this embedding with an embedding of a grid into the line which is the enumeration level by level. More formally if a node u is embedded to level i and a node v is embedded to level j and $i < j$ then the resulting number of u on the line is smaller than the number of v , but if two nodes are embedded to the same level, we give no guarantee.

Lemma 7. Any graph mapped into the ladder graph by the quasi-correct embedding described above can be mapped onto the line level by level with the property that any pair of adjacent nodes are embedded at the distance of at most five.

Assume, we are given a tree T that can be embedded into $Ladder_n$. Furthermore, there are two special nodes in the tree: one is marked as R (right) and another one is marked as L (left). It is known that there exists a correct embedding of T into $Ladder_n$ with R being the rightmost node, meaning no node is embedded more to the right or to the same level, and L being the leftmost node.

We now describe how to obtain a quasi-correct embedding of T onto $Ladder_N$ with R being the rightmost node and L being the leftmost one while L is mapped to $ImageL$ — some node of the $Ladder_N$. Moreover, our embedding obeys the following invariant.

Invariant 1 (Septum invariant) For each inner simple-graph, its foot and its head are embedded to the same level and no other node is embedded to that level.

We embed a path between L and R simply horizontally and then we orient line-graphs connected to it in a way that they do not violate our desired invariant. It can be shown that it is always possible if T can be embedded onto $Ladder_n$. The pseudocode is in Appendix Algorithm 1.

Suppose now that not all information, such as R , L , and $ImageL$, is provided. We explain how we can embed a tree T . We first get the *trunk core* of the given tree. This can be

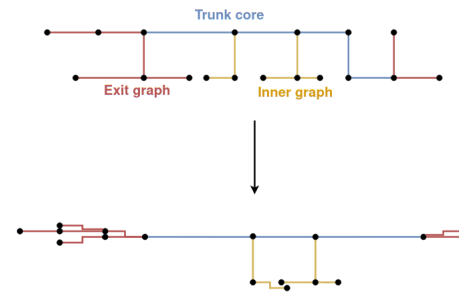


Fig. 4: Example of a quasi-correct embedding

done by following the definition. Now the idea would be to first embed the trunk core and its inner line-graphs using a tree embedding presented earlier with R and L to be the ends of the trunk core. Then, we embed exit-graphs strictly horizontally “away” from the trunk core. That means, that the hands of exit-graphs that are connected to the right of the trunk core are embedded to the right, and the hands of those exit-graphs that are connected to the left of the trunk core are embedded to the left. An example of the quasi-correct embedding is shown in Figure 4.

If a tree does not have a trunk core, then its structure is quite simple (in particular it has no more than two nodes of degree three). Such a tree can be embedded without conflicts. The pseudocode appears in Appendix Algorithm 2.

Cycle embedding Now, we show how to embed a cycle into $Ladder_N$. First, we give some important definitions and lemmas.

Definition 12. A maximal cycle C of a graph G is a cycle in G that cannot be enlarged, i.e., there is no other cycle C' in G such that $V(C) \subsetneq V(C')$.

Definition 13. Consider a graph G and a maximal cycle C of G . A whisker W of C is a line graph inside G such that: 1) $V(W) \neq \emptyset$ and $V(W) \cap V(C) = \emptyset$. 2) There exists only one edge between the cycle and the whisker (w, c) for $w \in V(W)$ and $c \in V(C)$. Such c is called a foot of W . The nodes of W are enumerated starting from w . 3) W is maximal, i.e., there is no W' in G such that W' satisfies previous properties and $V(W) \subsetneq V(W')$.



Fig. 5: Cycle and its Whiskers.

Definition 14. Suppose we have a graph G that can be correctly embedded into $Ladder_n$ by φ and a cycle C in G . Whiskers W_1 and W_2 of C are called adjacent (or neighboring) for the embedding φ if $\forall i \leq \min(|V(W_1)|, |V(W_2)|)$ $(\varphi(W_1[i]), \varphi(W_2[i])) \in E(Ladder_n)$.

Lemma 8. Suppose we have a graph G that can be correctly embedded into $Ladder_n$ and there exists a maximal cycle C in G with at least 6 vertices with two neighbouring whiskers W_1 and W_2 of C , i.e., $(foot(W_1), foot(W_2)) \in E(G)$. Then, W_1 and W_2 are adjacent in any correct embedding of G into $Ladder_N$.

Definition 15. Assume we have a graph G and a maximal cycle C of length at least 6. The frame for C is a subgraph of G induced by vertices of C and $\{W_1[i], W_2[i] \mid i \leq \min(|V(W_1)|, |V(W_2)|)\}$ for each pair of adjacent whiskers W_1 and W_2 . Adding all the edges $\{(W_1[i], W_2[i]) \mid i \leq \min(|V(W_1)|, |V(W_2)|)\}$ for each pair of adjacent whiskers W_1 and W_2 makes a frame completed.



Fig. 6: Cycle, its frame, and edges (dashed) to make the frame completed

Given a cycle C of length at least six and its special nodes $L, R \in V(C)$, we construct a correct embedding of C into $Ladder_N$ with $level\langle L \rangle \leq level\langle u \rangle \leq level\langle R \rangle \forall u \in V(C)$, while L is mapped into the node $ImageL$.

We first check if it is possible to satisfy the given constraints of placing the L node to the left and a R node to the right. If it is indeed possible, we place L to the desired place $ImageL$ and then we choose an orientation (clockwise or counterclockwise) following which we could embed the rest of the nodes, keeping in mind that R must stay on the rightmost level. The pseudocode appears in Appendix Algorithm 3.

Now, suppose that not all information, such as R , L , and $ImageL$, is provided. We reduce this problem to the case when the missing variables are known. This subtlety might occur since there are inner edges in the cycle. In this case, we choose missing L/R more precisely in order to embed an inner edge vertically. For more intuition, please see Figures 7a and 7b. A dashed line denotes an inner edge. The pseudocode appears in the Appendix (Algorithm 4).



Fig. 7: Cycle embeddings.

Embedding a connected component of the demand graph Combining the previous results, we can now explain how to embed onto $Ladder_N$ a connected component S that can be embedded onto $Ladder_n$.

Definition 16. *By the cycle-tree decomposition of a graph G we mean a set of maximal cycles $\{C_1, \dots, C_n\}$ of G and a set of trees $\{T_1, \dots, T_m\}$ of G such that*

- $\bigcup_{i \in [n]} V(C_i) \cup \bigcup_{i \in [m]} V(T_i) = V(G)$
- $V(C_i) \cap V(C_j) = \emptyset \quad \forall i \neq j$
- $V(T_i) \cap V(T_j) = \emptyset \quad \forall i \neq j$
- $V(T_i) \cap V(C_j) = \emptyset \quad \forall i \in [m], j \in [n]$
- $\forall i \neq j \quad \forall u \in V(T_i) \quad \forall v \in V(T_j) \quad (u, v) \notin E(G)$

We start with an algorithm on how to make a cycle-tree decomposition of S assuming no uncompleted frames. To obtain a cycle-tree decomposition of a graph: 1) we find a maximal cycle; 2) we split the graph into two parts by logically removing the cycle; 3) we proceed recursively on those parts, and, finally, 4) we combine the results together maintaining the correct order between cycle and two parts (first, the result for one part, then the cycle, and then the result for the second part). Since we care about the order of the parts, we say that it is a *cycle-tree decomposition chain*. The decomposition pseudocode appears in the Appendix Algorithm 5.

We describe how to obtain a quasi-correct embedding of S . We preprocess S : 1) we remove one edge from cycles of size four; 2) we complete uncompleted frames with vertical edges. Then, we embed parts of S from the cycle-tree decomposition chain one by one in the relevant order using the corresponding algorithm (either for a cycle or for a tree embedding) making sure parts are glued together correctly. The pseudocode appears in Appendix Algorithm 6.

4.2 Online quasi-embedding

In the previous subsection, we presented an algorithm on how to quasi-embed a static graph. Now, we will explain how to operate when the requests are revealed in an online manner. The full version of the algorithm is presented in Appendix B.2.

There are two cases: a known edge is requested or a new edge is revealed. In the first case the algorithm does nothing since we already know how to quasi-correctly embed the current graph and, thus, we already can embed into the line network with constant bandwidth. Thus, further, we will consider only the second case.

We describe how one should change the embedding of the graph after the processing of a request in an online scenario. At each moment some edges of the demand graph $Ladder_n$ are already revealed, forming connected components. After an edge reveal we should reconfigure the target line graph. For that, instead of line reconfiguration we reconfigure our embedding to $Ladder_N$ that is then embedded to the line level by level and introduces a constant factor. So, we can consider the reconfiguration only of $Ladder_N$ and forget about the target line graph at all. When doing the reconfiguration of an embedding we want to maintain the following invariants:

1. The embedding of any connected component is quasi-correct.
2. For each tree in the cycle-tree decomposition its embedding respects Septum invariant 1.
3. There are no maximal cycles of length 4.
4. Each cycle frame is completed with all “vertical” edges even if they are not yet revealed.
5. There are no conflicts with cycle nodes, i.e., each cycle node is the only node mapped to its image in the embedding to $Ladder_N$.

For each newly revealed edge there are two cases: either it connects two nodes from one connected component or not. We are going to discuss both of them.

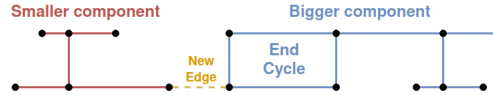
Edge in one component The pseudocode appears in Appendix Algorithm 8. If the new edge is already known or it forms a maximal cycle of length four, we simply ignore it. Otherwise, it forms a cycle of length at least six, since two connected nodes are already in one component. We then perform the following steps:

1. Get the completed frame of a (possibly) new cycle.
2. Logically “extract” it from the component and embed maintaining the orientation (not twisting the core that was already embedded in some way).
3. Attach two components appeared after an extraction back into the graph, maintaining their relative order.

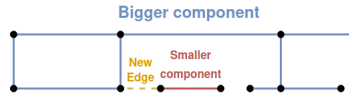
Edge between two components The pseudocode appears in Appendix Algorithm 9. In order to obtain an amortization in the cost, we always “move” the smaller component to the bigger one. Thus, the main question here is how to glue a component to the existing embedding of another component. The idea is

to consider several cases of where the smaller component will be connected to the bigger one. There are three possibilities:

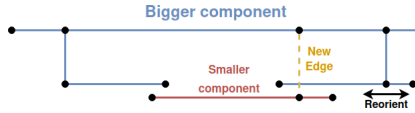
1. *It connects to a cycle node.* In this case, there are again two possibilities. Either it “points away” from the bigger component meaning that the cycle to which we connect is the one of the ends in the cycle-tree decomposition of the bigger component. Here, we just simply embed it to the end of the cycle-tree decomposition while possibly rotating a cycle at the end.



Or, the smaller component should be placed somewhere between two cycles in the cycle-tree decomposition. Here, it can be shown that this small graph should be a line-graph, and we can simply add it as a whisker, forming a larger frame.



2. *It connects to a trunk core node of a tree in the cycle-tree decomposition.* It can be shown that in this case the smaller component again must be a line-graph. Thus, our only goal is to orient it and possibly two of its inner simple-graphs neighbours to maintain the Septum invariant 1 for the corresponding tree from the decomposition.



3. *It connects to an exit graph node of an end tree of the cycle-tree decomposition.* In this case, we straightforwardly apply a static embedding algorithm of this tree and the smaller component from scratch. Please, note that only the exit graphs of the end tree will be moved since the trunk core and its inner graphs will remain.

4.3 Complexity of the online embedding

Now, we calculate the cost of our online algorithm (a more detailed discussion on the cost of the algorithm appears at Appendix C.5): how many swaps we should do and how much we should pay for the routing requests. Recall that we first apply the reconfiguration and, then, the routing request.

We start with considering the routing requests. Their cost is $O(1)$ since they lie pretty close on the target line network, i.e., by no more than 12 nodes apart. This bound holds because the nodes are quasi-correctly embedded on $Ladder_N$,

two adjacent nodes at G are located not more than four levels apart (in the worst case, when we remove an edge of a cycle with length four) where each level of the quasi-correct embedding has at most three images of nodes of G . Thus, on the target line graph, if we enumerate level by level, the difference between any two adjacent nodes of G is at most 12.

Then, we consider the reconfiguration. We count the total cost of each case of the online algorithm before all the edges are revealed.

In the first case, we add an edge in one component. By that, either a new frame is created or some frame was enlarged. In both cases, only the nodes, that appear on some frame for the first time, are moved. Since, a node can be moved only once to be mapped on a frame and it is swapped at most $N = O(n)$ times to move to any position, the total cost of this type of reconfiguration is at most $O(n^2)$. Also, there are several adjustments that could be done: 1) the “old” frame can rotate by one node, and 2) possibly, we should flip the first inner-graphs of two components connected to the frame. In the first modification, each node at the frame can only be “rotated” once, thus, paying $O(n)$ cost in total. In the second modification, inner-graph can change orientation at most once in order to satisfy the Septum invariant (Invariant 1), thus, paying $O(n^2)$ cost in total — each node can move by at most $N = O(n)$.

In the second case, we add an edge in between two components. At first, we calculate the time spent on the move of the small component to the bigger one: each node is moved at most $O(\log n)$ times since the size of the component always grows at least two times, the number of swaps of a vertex is at most $N = O(n)$ to move to any place, thus, the total cost is $O(n^2 \log n)$. Secondly, there are two more modification types: 1) a rotation of a cycle, and 2) some simple-graphs can be reoriented. The cycle can be rotated only once, thus, we should pay at most $O(n)$ there. At the same time, each simple-graph can be reoriented at most once to satisfy the Septum invariant (Invariant 1), thus, the total cost is $O(n^2)$ for that type of a reconfiguration.

To summarize, the total cost of requests σ is $O(n^2 \log n)$ for the whole reconfiguration plus $O(|\sigma|)$ per requests. This matches the lower bound that was obtained for the line demand graph. The same result holds for any demand graph that is the subgraph of the ladder of size n .

Theorem 6. *The online algorithm for embedding the ladder demand graph of size n on the line graph has total cost $O(n^2 \log n + |\sigma|)$ for a sequence of communication requests σ .*

5 Conclusion

We presented methods for statically or dynamically re-embedding a ladder demand graph (or a subgraph of it) on a line, both in the offline and online case. As side results, we also presented how to embed a cycle demand graph and a meta-algorithm for a general demand graph. Our algorithms for the cycle and the ladder cases match the lower bounds. Our work is a first step towards a tight bound on dynamically re-embedding more generic demand graphs, such as arbitrary grids.

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A The algorithm for the Cycle

We start with the most simple generalization result — when the demand graph is the cycle on n vertices.

Theorem 3. *Suppose the demand graph is C_n . There is an algorithm such that the total cost spent on the migrations is $O(n^2 \log n)$ and each request is performed in $O(1)$. In particular, if the number of requests is $\Omega(n^2 \log n)$ each request has $O(1)$ amortized cost.*

Proof. The idea of the algorithm is to act as in the algorithm described in [2] for the list demand graph until revealed edges do not form a cycle. Once they do we perform a total reconfiguration enumerating nodes of C_n with

$$\begin{cases} i \rightarrow 2i - 1, & \text{if } i \leq \lfloor \frac{n}{2} \rfloor \\ i \rightarrow 2(n - i + 1), & \text{otherwise} \end{cases}$$

so, for each pair of adjacent nodes the difference of their numbers is at most 2. The enumeration can be seen on Figure 8.

More formally. Let $G_i = (V, E_i)$ — the demand graph after i requests, and $G_0 = (V, \emptyset)$. We want to maintain the invariant that each G_i is embedded in a way that all adjacent nodes are at a distance of at most 2. Moreover, if there is a line subgraph of G_i then it is embedded as a line, i.e., the embedding preserves edges. We present an algorithm that maintains this invariant by induction.

This invariant holds for G_0 . We assume that the invariant holds for G_{i-1} and a new request σ_i arrives. If σ_i is already present in $E(G_{i-1})$ then the invariant holds, we do not reconfigure, and pay at most 2.

If now G_i is a cycle we perform a total reconfiguration with the enumeration with bandwidth 2 described above. For that we pay $O(n^2)$ that is less than $O(n^2 \log n)$, and, thus, our complexity lies inside our bounds. Note that once G_i becomes a cycle we need no further reconfigurations since all the edges are known and the invariant is maintained.

The last case is when σ_i is a new edge and G_i still consists of several connected components. We use the algorithm presented in [2]. σ_i connects two different connected components, say L_1 and L_2 forming a new list subgraph L . Suppose that $|V(L_1)| \leq |V(L_2)|$. Our strategy would be to “drag” L_1 towards L_2 that is if $L_1 = \{u_1, \dots, u_l\}$, $V(L_2) = \{v_1, \dots, v_k\}$, $\sigma_i = (v_k, u_1)$. By the invariant L_2 is embedded with v_p at $q + p$ for some q and we want nodes of L_1 to be embedded with $u_p \rightarrow q + k - 1 + p$. So, we bring each node of L_1 to its position performing required number of swaps. Note,

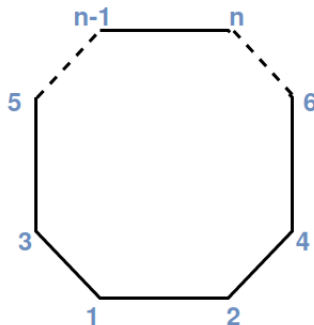


Fig. 8: Cycle enumeration with Bandwidth 2

that this reconfiguration brings the embedding that supports the invariant. Now, we analyze the cost of the algorithm processing the requests.

Due to the invariant each request is served with a cost of at most 2. As for the reconfiguration cost: a node can move a distance $\Theta(n)$ during processing one request and it moves no more than $O(\log n)$ times: we either form a cycle or merge two components. Thus, the total cost does not exceed $O(n^2 \log n)$.

B Full algorithm for the ladder

B.1 Static quasi-embedding

Tree embedding Assume, we are given a tree T that can be embedded into $Ladder_n$. Furthermore, there are two marked nodes in the tree: one is marked *right* and the *left*. It is known that there is a correct embedding of T with *right* being the right-most node, meaning no node is embedded higher or to the same level, and *left* being the left-most node.

We now describe how to obtain a quasi-correct embedding of T with *right* being the right-most node and *left* being the left-most one and *left* mapped to *leftImage*. Moreover, this embedding obeys the following invariant:

Invariant 2 (Septum invariant) *For each inner simple-graph its foot and its head are embedded to the same level and no other node is embedded to that level.*

We embed the *left* – *right* path strictly vertically and then we orient line-graphs connected to it in the way that they do not violate septum invariant.

See Algorithm 1.

Algorithm 1 Left-Right tree embedding

```

procedure RIGHTLEFTTREEEMBEDDING( $T, left, right, leftImage$ )
   $P \leftarrow$  path from  $left$  to  $right$ 
   $leftSide \leftarrow side(leftImage)$ 
   $leftLevel \leftarrow level\langle leftImage \rangle$ 
  Embed  $P[i] \rightarrow level(leftLevel - 1 + i)[leftSide]$ 
   $L \leftarrow$  line-graphs connected to  $P$ 
   $Septa \leftarrow \{level\langle foot(l) \rangle \mid l \in L\} \cup \{level\langle left \rangle, level\langle right \rangle\}$ 
  for  $l \in L$  do
     $i \leftarrow level\langle foot(l) \rangle$ 
    Embed  $head(l) \rightarrow level(i)[other(leftSide)]$ 
    Orient  $l$  ensuring no nodes of  $l \setminus \{head(l)\}$  are embedded to any of levels from
   $Septa$ 

```

We now proceed with an embedding of a tree T where *right*, *left* and *leftImage* might or might not be given. In the case the variable is not given we denote its value with None.

The idea here would be to first embed the trunk core and its inner line-graphs using Left-Right tree embedding and then to embed exit-graphs strictly vertically "away" from the trunk core. That means that the hands of exit-graphs that are connected to to the right of the trunk core are embedded increasingly and hands of those exit-graphs which are connected to the left of the trunk core are embedded decreasingly.

If a tree does not have a trunk core, that means that its structure is rather simple (in particular it has no more than two nodes of degree three), so we do not care about conflicts.

See Algorithm 2.

Algorithm 2 Tree quasi-correct embedding

```

procedure TREEQUASICORRECTEMBEDDING( $T$ ,  $left$ ,  $right$ ,  $leftImage$ )
  if  $leftImage = \text{None}$  then
     $leftImage \leftarrow level(1)[1]$ 
  if  $(left \neq \text{None}) \wedge (right \neq \text{None})$  then
    LeftRightTreeEmbedding( $T$ ,  $left$ ,  $right$ ,  $leftImage$ )
  return
  else if  $(left \neq \text{None}) \wedge (right = \text{None})$  then
    if  $T$  has a trunk core then
       $u, v \leftarrow$  ends of a trunk core
      if  $u$  between  $v$  and  $left$  then
         $trunkRight \leftarrow v$ 
      else
         $trunkRight \leftarrow u$ 
       $et \leftarrow$  exit-graphs connected to  $trunkRight$ 
       $S' \leftarrow S \setminus et$ 
      LeftRightTreeEmbedding( $S'$ ,  $left$ ,  $trunkRight$ ,  $leftImage$ )
      for  $e \in et$  do
         $lvl \leftarrow level\langle trunkRight \rangle$ 
         $side \leftarrow side(trunkRight)$ 
        Embed  $head(e) \rightarrow level(lvl + 1)[side]$ 
        for  $h \in hands(e)$  do
          for  $i \in [length(h)]$  do
            Embed  $h[j] \rightarrow level(lvl + 1 + i)[side]$ 
    else
       $right \leftarrow$  arbitrary node of degree 1
      LeftRightTreeEmbedding( $T$ ,  $left$ ,  $right$ ,  $leftImage$ )
  else if  $(left = \text{None}) \wedge (right \neq \text{None})$  then
    if  $T$  has a trunk core then
       $u, v \leftarrow$  ends of a trunk core
      if  $u$  between  $v$  and  $right$  then
         $trunkLeft \leftarrow v$ 
      else
         $trunkLeft \leftarrow u$ 

```

```

     $eb \leftarrow$  exit-graphs connected to  $trunkLeft$ 
     $S' \leftarrow S \setminus eb$ 
     $leftImageLevel \leftarrow level\langle leftImage \rangle$ 
     $leftImageSide \leftarrow side(leftImage)$ 
     $vShift \leftarrow \max_{e \in eb} \max_{h \in hands(e)} length(h)$ 
     $trunkLeftImage \leftarrow level(leftImageLevel + vShift)[leftImageSide]$ 
    LeftRightTreeEmbedding( $S'$ ,  $left$ ,  $right$ ,  $trunkLeftImage$ )
for  $e \in eb$  do
     $lvl \leftarrow level\langle trunkLeft \rangle$ 
     $side \leftarrow side(trunkLeft)$ 
    Embed  $head(e) \rightarrow level(lvl - 1)[side]$ 
    for  $h \in hands(e)$  do
    for  $i \in [length(h)]$  do
    Embed  $h[j] \rightarrow level(lvl - 1 - i)[side]$ 
else
     $left \leftarrow$  arbitrary node of degree 1
    LeftRightTreeEmbedding( $T$ ,  $left$ ,  $right$ ,  $leftImage$ )
else
if  $T$  has a trunk core then
     $trunkRight$ ,  $trunkLeft \leftarrow$  ends of a trunk core
     $et \leftarrow$  exit-graphs connected to  $trunkRight$ 
     $eb \leftarrow$  exit-graphs connected to  $trunkLeft$ 
     $S' \leftarrow S \setminus et \setminus eb$ 
     $leftImageLevel \leftarrow level\langle leftImage \rangle$ 
     $leftImageSide \leftarrow side(leftImage)$ 
     $vShift \leftarrow \max_{e \in eb} \max_{h \in hands(e)} length(h)$ 
     $trunkLeftImage \leftarrow level(leftImageLevel + vShift)[leftImageSide]$ 
    LeftRightTreeEmbedding( $S'$ ,  $left$ ,  $right$ ,  $trunkLeftImage$ )
    for  $e \in et$  do
     $lvl \leftarrow level\langle trunkRight \rangle$ 
     $side \leftarrow side(trunkRight)$ 
    Embed  $head(e) \rightarrow level(lvl + 1)[side]$ 
    for  $h \in hands(e)$  do
    for  $i \in [length(h)]$  do
    Embed  $h[j] \rightarrow level(lvl + 1 + i)[side]$ 
    for  $e \in eb$  do
     $lvl \leftarrow level\langle trunkLeft \rangle$ 
     $side \leftarrow side(trunkLeft)$ 
    Embed  $head(e) \rightarrow level(lvl - 1)[side]$ 
    for  $h \in hands(e)$  do
    for  $i \in [length(h)]$  do
    Embed  $h[j] \rightarrow level(lvl - 1 - i)[side]$ 
else

```

$right, left \leftarrow$ furthest nodes of degree 1
 $P \leftarrow$ path from $left$ to $right$
 Embed P strictly monotone placing $left$ to $leftImage$
if there is a line-graph left **then**
 $l \leftarrow$ left line-graph
 $(u, v) \leftarrow (u, v) \in E(T)$ s.t. $u \in P, v \in l$
 Embed v to the same level, opposite side to u .
 Embed l to the opposite side to the side of P preserving connectivity and correctness

Cycle embedding Given a cycle C of length ≥ 6 and nodes $left, right \in V(C)$ we construct a correct embedding of C into $Ladder_\infty$ with $level\langle left \rangle \leq level\langle u \rangle \leq level\langle right \rangle \forall u \in V(C)$ and $left$ placed to $leftImage$.

For convenience we assume that for every node v there is a local consecutive numeration starting at v . The number of node $u \in V(C)$ in this numeration is referenced with $number_v(u)$. The node with number i in local numeration of v is referenced with $C_v[i]$.

We first check if it is possible to satisfy the given constraints of placing the $left$ node to left and a $right$ node to the right. If it is indeed possible, we place $left$ to desired place and then choose an orientation (clockwise or counterclockwise) following which we would embed the rest of the nodes, keeping in mind that $right$ must stay on the highest level. See Algorithm 3.

Algorithm 3 Left-Right cycle embedding

procedure LEFTRIGHTCYCLEEMBEDDING($C, left, right, leftImage$)
 $h \leftarrow \frac{length(C)}{2}$
 Ensure: $number_{left}(right) \in \{h, h + 1, h + 2\}$
 $leftLevel \leftarrow level\langle leftImage \rangle$
 $leftSide \leftarrow side(leftImage)$
 if $(number_{left}(right) = h) \vee (number_{left}(right) = h + 1)$ **then**
 for $i \in [h]$ **do**
 Embed $C_{left}[i] \rightarrow level(leftLevel + i - 1)[leftSide]$
 for $i \in [h]$ **do**
 Embed $C_{left}[h + i] \rightarrow level(leftLevel + h - i)[other(leftSide)]$
 else
 Embed $left \rightarrow leftImage$
 for $i \in [h]$ **do**
 Embed $C_{left}[i + 1] \rightarrow level(leftLevel + i - 1)[other(leftSide)]$
 for $i \in [h - 1]$ **do**
 Embed $C_{left}[h + 1 + i] \rightarrow level(leftLevel + h - i)[leftSide]$

Now, suppose that not all information, such as $right$, $left$, and $LeftImage$, is provided. We will reduce this problem to the case when the missing variables are known. Though the subtlety might occur due to the fact that there are inner

edges in the cycle. In this case we choose missing *left/right* more precisely in order to embed inner edge vertically. See Algorithm 4.

Algorithm 4 Cycle embedding

```

procedure CYCLEEMBEDDING( $C, left, right, leftImage$ )
   $h \leftarrow \frac{\text{length}(C)}{2}$ 
  if ( $left \neq \text{None}$ )  $\wedge$  ( $right \neq \text{None}$ ) then
    Ensure:  $\text{number}_{left}(right) \in \{h, h + 1, h + 2\}$ 
  if  $leftImage = \text{None}$  then
     $leftImage \leftarrow \text{level}(1)[1]$ 
  if ( $left = \text{None}$ )  $\wedge$  ( $right = \text{None}$ ) then
     $left \leftarrow$  arbitrary node of  $C$ 
    if  $C$  has an inner edge then
      Choose  $right$  out of  $\{C_{left}[h], C_{left}[h + 2]\}$  to respect the inner edge
    else
      Choose  $right$  out of  $\{C_{left}[h], C_{left}[h + 2]\}$  arbitrary
  else if ( $left \neq \text{None}$ )  $\wedge$  ( $right = \text{None}$ ) then
    if  $C$  has an inner edge then
      Choose  $right$  out of  $\{C_{left}[h], C_{left}[h + 2]\}$  to respect the inner edge
    else
      Choose  $right$  out of  $\{C_{left}[h], C_{left}[h + 2]\}$  arbitrary
  else if ( $left = \text{None}$ )  $\wedge$  ( $right \neq \text{None}$ ) then
    if  $C$  has an inner edge then
      Choose  $left$  out of  $\{C_{left}[h], C_{left}[h + 2]\}$  to respect the inner edge
    else
      Choose  $left$  out of  $\{C_{left}[h], C_{left}[h + 2]\}$  arbitrary
  LeftRightCycleEmbedding( $C, left, right, leftImage$ )

```

Component embedding Right now we explain on how to embed onto $Ladder_N$ a connectivity component S that can be embedded onto $Ladder_n$.

We start with an algorithm on how to make a cycle-tree decomposition chain of S assuming no uncompleted frames. To obtain a cycle-tree decomposition of a graph: 1) we find a maximal cycle; 2) we split the graph into two parts by logically removing the cycle; 3) we proceed recursively on those parts, and, finally, 4) we combine the results together maintaining the correct order of the chain components. See the Algorithm 5.

Now, we describe how to obtain a quasi-correct embedding of S . We preprocess S : 1) we remove one edge from cycles of size four; 2) we complete uncompleted frames with vertical edges. After this preprocessing, we embed parts of S from the cycle-tree decomposition chain one by one in the relevant order using the corresponding algorithm (either for a cycle or for a tree embedding) making sure parts are glued together correctly.

Algorithm 5 Cycle-Tree decomposition chain

```

function CYCLETREEDECOMPOSITIONCHAIN( $S$ )
Ensure:  $S$  has no uncompleted frames
  if  $S$  is empty then
    return []
  else if  $S$  is a tree then
    return [ $S$ ]
  else
     $C \leftarrow$  arbitrary maximal cycle in  $S$ 
     $S_1, S_2 \leftarrow$  connectivity components of  $S \setminus C$ 
     $C_1 \leftarrow$  CycleTreeDecompositionChain( $S_1$ )
     $C_2 \leftarrow$  CycleTreeDecompositionChain( $S_2$ )
    if  $C_2$  is empty then
      return [ $C$ ] +  $C_1$ 
    else
      if  $\exists u \in V(C_2[0]), v \in V(C), s.t. (u, v) \in E(S)$  then
        return  $C_1$  + [ $S$ ] +  $C_2$ 
      else
        return  $C_2$  + [ $S$ ] +  $C_1$ 

```

As before, we have additional variables *left*, *right* and *leftImage* which might or might not be given.

Algorithm 6 Connectivity component quasi-correct embedding

```

procedure PREPROCESS( $S$ )
   $C \leftarrow$  maximal cycles of length 4 in  $S$ 
  for  $c \in C$  do
    remove arbitrary edge of  $c$  from  $S$ 
   $F \leftarrow$  cycle frames in  $S$ 
  for  $f \in F$  do
    complete  $F$ 
procedure COMPONENTEMBEDDINGLEFTFIXED( $S, left, right, leftImage$ )
  if  $S$  is a tree then
    TreeQuasiCorrectEmbedding( $S, left, right, leftImage$ )
    return
  if  $S$  is a cycle then
    CycleEmbedding( $S, left, right, leftImage$ )
    return
  Preprocess( $S$ )
   $C \leftarrow$  CycleTreeDecompositionChain( $S$ )
  if  $left \neq \text{None}$  then
    Reverse  $C$  in the way that  $left \in C[1]$ 
  if  $right \neq \text{None}$  then
    Reverse  $C$  in the way that  $right \in C[\text{length}(C)]$ 
  for  $i \in [\text{length}(C)]$  do

```

```

if  $i = 1$  then  $u, v \leftarrow (u, v) \in E(S)$ , s.t.  $(u \in V(C[1])) \wedge (v \in V(C[i+1]))$ 
  if  $C[1]$  is a tree then
     $cur \leftarrow C[1] \cup (u, v)$ 
    TreeQuasiCorrectEmbedding( $cur$ ,  $left$ ,  $v$ ,  $leftImage$ )
  else
    CycleEmbedding( $C[1]$ ,  $left$ ,  $u$ ,  $leftImage$ )
else if  $i = length(C)$  then
   $u, v \leftarrow (u, v) \in E(S)$ , s.t.  $(u \in V(C[i-1])) \wedge (v \in V(C[i]))$ 
   $leftLevel \leftarrow level\langle u \rangle + 1$ 
   $leftSide \leftarrow side(u)$ 
   $localLeftImage \leftarrow level(leftLevel)[leftSide]$ 
  if  $C[i]$  is a cycle then
    CycleEmbedding( $C[i]$ ,  $v$ ,  $right$ ,  $localLeftImage$ )
  else
    TreeQuasiCorrectEmbedding( $C[i]$ ,  $v$ ,  $right$ ,  $localLeftImage$ )
else
   $u_1, v_1 \leftarrow (u, v) \in E(S)$ , s.t.  $(u \in V(C[i-1])) \wedge (v \in V(C[i]))$ 
   $leftLevel \leftarrow level\langle u_1 \rangle + 1$ 
   $leftSide \leftarrow side(u_1)$ 
   $localLeftImage \leftarrow level(leftLevel)[leftSide]$ 
   $u_2, v_2 \leftarrow (u, v) \in E(S)$ , s.t.  $(u \in V(C[i])) \wedge (v \in V(C[i+1]))$ 
  if  $C[i]$  is a cycle then
    CycleEmbedding( $C[i]$ ,  $v_1$ ,  $u_2$ ,  $localLeftImage$ )
  else
     $cur \leftarrow C[i] \cup (u_2, v_2)$ 
    TreeQuasiCorrectEmbedding( $C[i]$ ,  $v_1$ ,  $v_2$ ,  $localLeftImage$ )

```

We finally notice that having a procedure to embed a component with a fixed $leftImage$ it is easy to obtain a procedure which embeds with $rightImage$ fixed. We simply apply the "left" procedure and then flip the result.

Algorithm 7 Component embedding right fixed

```

procedure COMPONENTEMBEDDINGRIGHTFIXED( $S$ ,  $left$ ,  $right$ ,  $rightImage$ )
  ComponentEmbeddingLeftFixed( $S$ ,  $right$ ,  $left$ ,  $rightImage$ )
  Flip the image of  $S$  over horizontal axis maintaining the position of  $right$ 

```

B.2 Dynamic algorithm

We describe how one should change the embedding of the graph after the processing of a request in an online scenario. At each moment we have some edges of a $Ladder_n$ already revealed forming connectivity components. After an edge reveal we should reconfigure the target line graph. For that, instead of line reconfiguration we reconfigure our embedding to $Ladder_N$ that is then embedded

to the line line by line and introduce some constant factor. So, we can consider the reconfiguration only of $Ladder_N$ and forget about the target line graph at all. When doing the reconfiguration of an embedding we want to maintain the following invariants:

1. The embedding of any connectivity component is quasi-correct.
2. For each tree in the cycle-tree decomposition its embedding respects Septum invariant 1.
3. There are no maximal cycles of length 4.
4. Each cycle frame is completed with all “vertical” edges even if they are not yet revealed.
5. There are no conflicts with cycle nodes, i.e., two nodes of a cycle do not map to same node of $Ladder_N$.

For each newly revealed edge there are two cases: either it connects two nodes from one connectivity component or not. We are going to discuss both of them.

Edge in one component If the new edge is already known or it forms a maximal cycle of length four, we simply ignore it. Otherwise, it forms a cycle of length at least six, since two connected nodes are already in one component.

We then perform the following steps:

1. Get the completed frame of a (possibly) new cycle.
2. Logically “extract” it from the component and embed maintaining the orientation (not twisting the core that was already embedded in some way).
3. Attach two components appeared after an extraction back into the graph, maintaining their relative order.

Algorithm 8 Process Edge In One Component

```

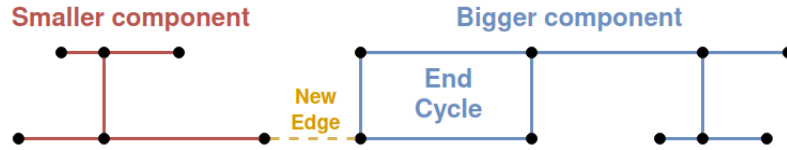
procedure PROCESSEDGEONECOMPONENT( $S, (u, v)$ )
  if Edge  $(u, v)$  already exists then
    return
  if Edge  $(u, v)$  forms a maximal cycle of length 4 then
    return
   $C \leftarrow$  maximal cycle containing  $u, v$ 
   $F \leftarrow$  completed frame of  $C$ 
   $S_1, S_2 \leftarrow$  connectivity components of  $S \setminus F$ 
   $u_1, v_1 \leftarrow u, v : u \in V(F), v \in V(S_1), (u, v) \in E(S)$ 
   $u_2, v_2 \leftarrow u, v : u \in V(F), v \in V(S_2), (u, v) \in E(S)$ 
  if  $level\langle u_1 \rangle > level\langle u_2 \rangle$  then
     $Swap(S_1, S_2), Swap(u_1, u_2), Swap(v_1, v_2)$ 
  CycleEmbedding( $F, u_1, u_2, None$ )
  ComponentEmbeddingTopFixed( $S_1, None, u_1, image(u_1)$ )
  ComponentEmbeddingBotFixed( $S_2, u_2, None, image(u_2)$ )

```

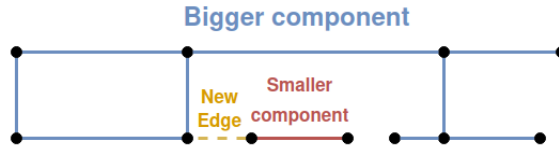
Edge between two components In order to obtain an amortization in the cost, we always “move” the smaller component to the bigger one. Thus, the main question here is how to glue a component to the existing embedding of another component.

The idea is to consider several cases of where the smaller component will be connected to the bigger one. There are three possibilities:

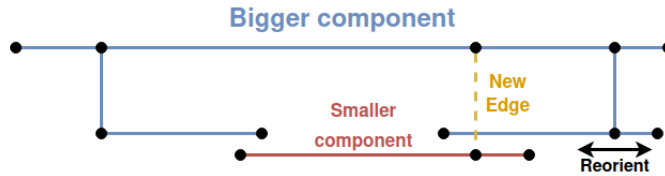
1. *It connects to a cycle node.* In this case there are again two possibilities. Either it “points away” from the bigger component meaning that the cycle to which we connect is the one of the ends in the cycle-tree decomposition of the bigger component. Here, we just simply embed it to the end of the cycle-tree decomposition while possibly rotating a cycle at the end.



Or, the smaller component should be placed somewhere between two cycles in the cycle-tree decomposition. Here, it can be shown that this small graph should be a line-graph, and we can simply add it as a whisker, forming a larger frame.



2. *It connects to a trunk core node of a tree in the cycle-tree decomposition.* It can be shown that in this case the smaller component again must be a line-graph. Thus, our only goal is to orient it and possibly two of its inner simple-graphs neighbours to maintain the Septum invariant 1 for the corresponding tree from the decomposition.



3. *It connects to an exit graph node of an end tree of the cycle-tree decomposition.* In this case, we straightforwardly apply a static embedding algorithm of this tree and the smaller component from scratch. Please, note that only the exit graphs of the end tree will be moved since the trunk core and its inner graphs will remain.

Algorithm 9 Process edge between two components

```

procedure ADDINNERWHISKER( $S, C, W, (u, v)$ )
   $S \leftarrow S \cup W \cup (u, v)$ 
   $F \leftarrow$  completed frame of  $C$ 
   $S_1, S_2 \leftarrow$  connectivity components of  $S \setminus F$ 
  if  $S_1$  is embedded above  $S_2$  then
     $Swap(S_1, S_2)$ 
   $s_1, t_1 \leftarrow s, t : s \in V(F), t \in V(S_1), (s, t) \in E(S)$ 
   $s_2, t_2 \leftarrow s, t : s \in V(F), t \in V(S_2), (s, t) \in E(S)$ 
  CycleEmbedding( $F, s_1, s_2, \text{None}$ )
  ComponentEmbeddingTopFixed( $S_1 \cup (s_1, t_1), \text{None}, s_1, image(s_1)$ )
  ComponentEmbeddingBotFixed( $S_2 \cup (s_2, t_2), s_2, \text{None}, image(s_2)$ )
procedure PROCESSEDEGETWOCOMPONENTS( $S_1, S_2, (u, v)$ )
  Ensure:  $u \in V(S_1), v \in V(S_2)$ 
  if  $V(S_1) < V(S_2)$  then
     $Swap(S_1, S_2), Swap(u, v)$ 
   $DC_1 \leftarrow CycleTreeDecomposition(S_1)$ 
  Reverse  $DC_1$  in a way  $DC[i]$  is embedded under  $DC[i + 1] \forall i$ 
   $A, i \leftarrow DC_1[i], i : u \in V(DC_1[i])$ 
  if  $A$  is a cycle then
    if  $length(DC_1) = 1$  then
      if  $A$  has an inner edge then
        if  $u$  is a top node then
           $bot \leftarrow$  arbitrary bottom node of  $A$ 
          ComponentEmbeddingBotFixed( $S_1 \cup S_2 \cup (u, v), bot, \text{None},$ 
None)
        else
           $top \leftarrow$  arbitrary top node of  $A$ 
          ComponentEmbeddingTopFixed( $S_1 \cup S_2 \cup (u, v), \text{None}, top,$ 
None)
        else
          ComponentEmbeddingBotFixed( $S_1 \cup S_2 \cup (u, v), \text{None}, \text{None},$ 
None)
      else if  $i = 1$  then
         $p, q \leftarrow p, q : p \in V(DC_1[i]), q \in V(DC_1[i + 1]), (p, q) \in E(S_1)$ 
        if  $(u, p) \in E(S_1)$  then AddInnerWhisker( $S_1, A, S_2, (u, v)$ )
        else
          if  $u$  is not a bottom node of  $A$  then
            Flip  $A$  over diagonal containing  $p$ 
            ComponentEmbeddingTopFixed( $S_2, \text{None}, u, image(u)$ )
          else if  $i = length(DC_1)$  then
             $p, q \leftarrow p, q : p \in V(DC_1[i]), q \in V(DC_1[i - 1]), (p, q) \in E(S_1)$ 
            if  $(u, p) \in E(S_1)$  then AddInnerWhisker( $S_1, A, S_2, (u, v)$ )

```

```

else
  if  $u$  is not a top node of  $A$  then
    Flip  $A$  over diagonal containing  $p$ 
    ComponnetEmbeddingBotFixed( $S_2, u, \text{None}, \text{image}(u)$ )
  elseAddInnerWhisker( $S_1, A, S_2, (u, v)$ )
if  $A$  is a tree then
  if  $u \in$  extended trunk core of  $A$  then
    Embed  $v \rightarrow \text{opposite}(u)$ 
     $l_1, l_2 \leftarrow u$  neighbouring inner simple-graphs
    Orient  $S_2, l_1, l_2$  to maintain Septum invariant in  $A$ 
  else if  $u \in$  inner simple-graph then
     $S_1 \leftarrow S_1 \cup S_2 \cup (u, v)$ 
     $l \leftarrow$  inner simple graph containing  $u$ 
    Orient  $l$  to maintain Septum invariant in  $A$ 
  else if  $u \in$  exit-graph then
    if  $i = 1$  then
      if  $\text{length}(DC_1) = 1$  then
         $p \leftarrow$  arbitrary highest node of  $S_1$ 
         $q \leftarrow$  additional temporary node
      else
         $p, q \leftarrow p, q : p \in V(DC_1[i]), q \in V(DC_1[i+1]), (p, q) \in E(S_1)$ 
        ComponentEmbeddingTopFixed( $A \cup S_2 \cup (p, q), \text{None}, q, \text{image}(q)$ )
      else if  $i = \text{length}(DC_1)$  then
         $p, q \leftarrow p, q : p \in V(DC_1[i]), q \in V(DC_1[i-1]), (p, q) \in E(S_1)$ 
        ComponentEmbeddingBotFixed( $A \cup S_2 \cup (p, q), q, \text{None}, \text{image}(q)$ )

```

C Proofs and Analysis

C.1 Strategy

At the very beginning there are no requests and we don't know any request-edges. Requests come one at a time, possibly, revealing new edges. Known edges form connectivity components which are all subgraphs of the request graph. Our strategy would be to maintain such enumeration σ of vertices that for each connectivity component S

$$\max_{(u,v) \in E(S)} |\sigma(u) - \sigma(v)| \leq 12 \quad (1)$$

We call this property of an enumeration the *proximity property*.

So, if we receive the request which was already known, we do nothing since the property persists. But if the new edge comes, we might perform a re-enumeration σ on the vertices to maintain the property.

C.2 Bandwidth of subgraphs

Definition 17. Consider two connected graphs S and G . The correct embedding of S into G is a mapping $\varphi : V(S) \rightarrow V(G)$ such that:

- φ is injective
- $(u, v) \in E(S) \rightarrow (\varphi(u), \varphi(v)) \in E(G)$

If φ is not injective, i.e. there are nodes u, v , s.t. $\varphi(u) = \varphi(v)$, we say that there is a conflict between u and v .

Lemma 2. For each subgraph S of a graph G , $\text{Bandwidth}(S) \leq \text{Bandwidth}(G)$.

Proof. Let φ be a correct embedding of S into G . And let σ be the enumeration on G with which the $\text{bandwidth}(G)$ is achieved.

Let U be the finite set of unique natural numbers. For $v \in U$ we define $\text{ord}_U(v) = |\{u \mid u \in U, u \leq v\}|$.

We now define the enumeration σ_S of S as follows:

$$\begin{aligned} U &= \{\sigma(\varphi(v)) \mid v \in S\} \\ \sigma_S(v) &= \text{ord}_U(\sigma(\varphi(v))) \end{aligned}$$

We state that $\max_{(u,v) \in E(S)} |\sigma_S(u) - \sigma_S(v)| \leq \text{bandwidth}(G)$. This follows from two facts:

- For every $(u, v) \in E(S)$

$$|\sigma(\varphi(u)) - \sigma(\varphi(v))| \leq \text{bandwidth}(G)$$

since $(\varphi(u), \varphi(v)) \in E(G)$

- If U is a set of unique natural numbers than for every $u, v \in U$

$$|\text{ord}_U(u) - \text{ord}_U(v)| \leq |u - v|$$

Then, for each edge $(u, v) \in E(S)$ we get the following inequalities:

$$\begin{aligned} |\sigma_S(u) - \sigma_S(v)| &= |\text{ord}_U(\sigma(\varphi(u))) - \text{ord}_U(\sigma(\varphi(v)))| \\ &\leq |\sigma(\varphi(u)) - \sigma(\varphi(v))| \leq \text{bandwidth}(G) \end{aligned}$$

We know that all the graphs that appear during the requests processing (revealing the edges) are subgraphs of $Ladder_n$. Thus, by Lemma 2 we conclude that their $\text{bandwidth} \leq 2$. And we can use the embedding function from this Lemma to enumerate each subgraph S of $Ladder_n$ with σ_S .

Remark 1. We do not need to worry about the top and bottom bounds of $Ladder_n$ when performing an embedding. In fact, we can perform an embedding of S into the $Ladder_\infty$ and, since S is connected and the embedding preserves connectivity, the whole image of S will be within some $Ladder_m$ (for $m \geq n$) which is enough to obtain a requested $\text{bandwidth} \leq 2$.

C.3 Connectivity component structure

Requests come with time possibly revealing new edges of a request graph and forming connectivity components which are subgraphs of the request graph.

One connectivity component can be decomposed into cycles and trees. Let us now provide some statements about tree and cycle embedding.

Tree embedding

Definition 6. Consider some correct embedding φ of a tree T into $Ladder_n$. Let $r = \arg \max_{v \in V(T)} level\langle\varphi(v)\rangle$ be the “rightmost” node of the embedding and $l = \arg \min_{v \in V(T)} level\langle\varphi(v)\rangle$ be the “leftmost” node of the embedding. The trunk of T is a path in T connecting l and r . The trunk of a tree T for the embedding φ is denoted with $trunk_\varphi(T)$.

Definition 7. Let T be a tree and φ be its correct embedding into $Ladder_n$. The level i of $Ladder_n$ is called occupied if there is a vertex $v \in V(T) : \varphi(v) \in level_{Ladder_n}(i)$.

Statement 1 For every occupied level i there is $v \in trunk_\varphi(T)$ such that $v \in level(i)$.

Proof. By the definition of the trunk, an image goes from the minimal occupied level to the maximal. It cannot skip a level since the trunk is connected and the correct embedding preserves connectivity.

The trunk of a tree in an embedding is an useful concept to define since the following holds for it.

Lemma 4. Let T be a tree correctly embedded into $Ladder_n$ by some embedding φ . Then, all the connected components in $T \setminus trunk_\varphi(T)$ are line-graphs.

Proof. Suppose that it is not true and then there should exist a subgraph S of T such that $V(S) \cap V(trunk_\varphi(T)) = \emptyset$ and S contains a node of degree three. Since there is a node of degree three in S we can state that there are two nodes of S , say u and v with the same level ($level\langle\varphi(v)\rangle = level\langle\varphi(u)\rangle$). But the image of the tree trunk passes through all occupied levels of the grid by Statement 1. Hence, either u or $v \in trunk_\varphi(T)$ which contradicts the assumption.

The bad thing about the trunk is that it depends on the embedding. And there can be several correct embeddings of the same tree giving different trunks. So, we introduce the concept of a *trunk core* which alleviates this issue. But at first, we prove some technical statements.

Statement 2 For the tree T , disregarding the correct embedding φ , the $trunk_\varphi(T)$ must pass through a node of degree three if it has no neighbours of degree three. If there are two adjacent nodes with degree three, the trunk must pass through at least one of them.

Proof. First, consider the case of a node with no neighbours of degree three. Let's call it a . To prove by contradiction we assume that the trunk does not pass through a . Let's call a 's neighbours b, c and d . W.l.o.g assume that

$$\varphi(a) = \text{level}(i)[1] \quad (2)$$

$$\varphi(b) = \text{level}(i-1)[1] \quad (3)$$

$$\varphi(c) = \text{level}(i)[2] \quad (4)$$

$$\varphi(d) = \text{level}(i+1)[1] \quad (5)$$

Since the trunk does not pass through a and by Statement 1 it passes through the level i it should pass through c . If c has degree one, the trunk contains one node from level i , and does not contain any node from $i+1$, thus, this trunk cannot contain the topmost node. If c has degree two, we say that its second neighbour is mapped to $\text{level}(i-1)[2]$. The case when it is mapped to $\text{level}(i+1)[2]$ is symmetric. But then trunk does not pass through the $i+1$ level which contradicts the Statement 1.

Now, coming to the case with two adjacent nodes of degree three, we have two adjacent nodes a and b of degree three. And let c, d be the rest neighbours of a and e, f be the rest neighbours of b . If a and b are embedded to the same level, then by Statement 1 the trunk passes through at least one of them. Suppose now that a and b are on different levels, say

$$\varphi(a) = \text{level}(i)[1] \quad (6)$$

$$\varphi(b) = \text{level}(i+1)[1] \quad (7)$$

$$\varphi(c) = \text{level}(i)[2] \quad (8)$$

$$\varphi(d) = \text{level}(i-1)[2] \quad (9)$$

$$\varphi(e) = \text{level}(i+1)[2] \quad (10)$$

$$\varphi(f) = \text{level}(i+2)[1] \quad (11)$$

Since the edge (a, b) is a bridge between two connected components of a tree and the trunk contains nodes from both components the trunk should pass through the edge (a, b) , so it passes through both a and b .

Lemma 5. *For the tree T for each node v of degree three (except for maximum two of them) we can verify in polynomial time if for any correct embedding φ trunk $_{\varphi}(T)$ passes through v or not.*

Proof. We call a pair of adjacent nodes of degree three “paired” nodes. We call a node of degree three with no neighbours of degree three “single”.

If the tree contains not more than two nodes of degree three, the statement is trivial. So, we suppose that there exist at least three nodes of degree three.

The trunk passes through the single nodes by Statement 2. Thus we are interested in paired nodes. Consider such pair. Let's call its nodes a and b . By the Statement 2 we know that either a or b is in the trunk.

By the assumption there exist either another single node or other paired nodes. If there is a single node, let's call it c , we know that it is in the trunk. c

is reachable from a and b and since we have tree either a is on path from b to c or b is on path from a to c . W.l.o.g. assume b is on a path from a to c . But this implies that b is in the trunk, because if not, a is, and, thus, there are two paths from a to c — the trunk and the one containing b . Thus, we have a cycle, which is impossible since we have a tree.

If there are no single nodes, there are paired nodes. We denote them with u and v . u and v are reachable from a , thus, w.l.o.g. we can assume that u is on the path from a to v . If now b is on the path from a to u , we have the following: $a \rightarrow b \rightsquigarrow u \rightarrow v$. By Statement 2 we know that the trunk must pass through either u or v . Denote the one the trunk passes through with c . We can reduce this case to the previous one, if we take any $c = u$ or $c = v$. Applying the same reasoning we deduce that b must be in the trunk.

Support nodes are the nodes of two types: either it is a single node, or it is a node that is located on the path between two other nodes with degree three. This Lemma shows that the support nodes appear in the trunk of every correct embedding.

We make a path P through support nodes. For any inner node of this path which is paired there is no chance for its pair to be in a trunk if it is not in P already, because the trunk is a path. So, the only uncertainty remains about at most one node in the pairs of end nodes.

Definition 18. Path P constructed in Lemma 5 is called trunk core. We denote this path for a tree T as $\text{trunkCore}(T)$. Note that it can be embedded into Ladder_n .

Definition 19. The embedding φ of a line-graph l on the grid is called monotone if the nodes $\varphi(l[i])$ and $\varphi(l[j])$ are on the same level of the grid only when they are adjacent on T .

Lemma 9. If a line graph is embedded preserving edges into Ladder_n with no self-intersections non-monotonically then one of the end-points shares a level with a node of a path it is not adjacent with.

Proof. Denote a line graph l . Let's say i is the smallest index such that $l[i]$ shares level with some other non-adjacent node $l[j]$, $|i - j| > 1$. W.l.o.g. let's assume that $l[i]$ is embedded to $\text{level}(k)[1]$. Since i was chosen the smallest $j > i$. Let us assume that $l[i - 1]$ is embedded to $\text{level}(k - 1)[1]$. Then, since $l[j]$ is embedded to $\text{level}(k)[2]$, $l[i + 1]$ is embedded into $\text{level}(k + 1)[1]$. We also state that $l[j - 1]$ is embedded to $\text{level}(k + 1)[2]$, since if it is embedded into $\text{level}(k - 1)[2]$, the path should go from $l[i + 1]$ to $l[j - 1]$ (note that $i + 1 < j - 1$) without passing through level k which is impossible. So we have the following embeddings:

$$l[i] \rightarrow \text{level}(k)[1] \tag{12}$$

$$l[i - 1] \rightarrow \text{level}(k - 1)[1] \tag{13}$$

$$l[j] \rightarrow \text{level}(k)[2] \tag{14}$$

$$l[j + 1] \rightarrow \text{level}(k - 1)[2] \tag{15}$$

It is easy to see that $l[j + 2]$ has no other options but to be embedded to $level(k - 2)[2]$. But then $l[i - 3]$ should be embedded to $level(k - 3)[1]$ and so on $l[j + t] \rightarrow level(k - t)[2]$ and $l[i - t] \rightarrow level(k - t)[2]$ in general. We now take $t = \min(i - 1, length(p) - j)$ so either $l[i - t]$ or $l[j + t]$ is an end nodes and they both exist. They share level, so the lemma is proved.

Lemma 10. *The trunk core of a tree T is always embedded in the monotone manner.*

Proof. Trunk core connects nodes of degree three which cannot be embedded with any other nodes of the trunk to the same level since then either a cycle appears or we obtain a conflict. Thus, by the Lemma 9 trunk core must be embedded monotonically.

From now on we assume that every mentioned tree can be embedded into the $Ladder_n$.

Definition 20. *Let T be a tree. All the connectivity components in $T \setminus trunkCore(T)$ are called simple-graphs of tree T .*

Lemma 6. *Simple-graphs of a tree T are line-graphs.*

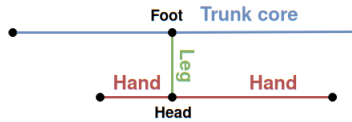
Proof. Note that all the nodes of degree three in T are either in the trunk core or they are adjacent to the trunk core. Hence after removing the nodes of the trunk core no nodes of degree three are left and, thus, all the graphs left are line-graphs.

Definition 9. *The edge between a simple-graph and the trunk core is called a leg.*

The end of a leg in the simple-graph is called a head of the simple-graph.

The end of a leg in the trunk core is called a foot of the simple-graph.

If you remove the head of a simple-graph and it falls apart into two connected components, such simple-graph is called two-handed and those parts are called its hands. Otherwise, the graph is called one-handed, and the sole remaining component is called a hand. If there are no nodes in the simple-graph but just a head it is called zero-handed.



Definition 10. *A simple-graph connected to the end nodes of the trunk core is called exit-graph.*

Definition 10. *A simple-graphs connected to the inner nodes of the trunk core is called inner-graph.*

Please note that the next definition is about a much larger ladder $Ladder_N$ rather than $Ladder_n$. N should be approximately equal to $2 \cdot n$.

Definition 11. An embedding $\varphi : V(G) \rightarrow V(\text{Ladder}_N)$ of a graph G into Ladder_N is called quasi-correct if:

- $(u, v) \in E(G) \Rightarrow (\varphi(u), \varphi(v)) \in E(\text{Ladder}_N)$, i.e., images of adjacent vertices in G are adjacent in the grid.
- There are no more than **three** nodes mapped into each level of Ladder_N , i.e., the two grid nodes on each level are the images of no more than three nodes.

We might think of a quasi-correct embedding as an embedding into levels of the grid with no more than three nodes embedded to the same level. We then can compose this embedding with an embedding of a grid into the line which is the enumeration level by level. More formally if a node u is embedded to the level i and a node v is embedded to the level j and $i < j$ then the resulting number of u on the line is smaller than the number of v , but if two nodes are embedded to the same level, we give no guarantee.

Lemma 7. For any graph mapped into the ladder graph by the quasi-correct embedding as described above can be mapped onto the line level by level with the property that any pair of adjacent nodes are embedded at the distance of at most five.

Proof. Since two adjacent vertices are embedded to the levels with a number difference of at most 1, we can state that there are no more than 4 nodes between them in the line, since there are no more than 3 nodes per level.

Tree embedding strategy We start with the discussion on how to embed a tree with $|V(\text{trunkCore}(T))| \leq 1$. Such tree can have ≤ 3 nodes of degree three, since, otherwise, there are at least four nodes of degree three and the trunk core has at least two nodes:

- ≥ 2 single nodes. In this case, they are both in the trunk core.
- at least one single node and at least one paired nodes. In this case, one node from a pair and a single node are in the trunk core.
- at least two disjoint paired nodes. In this case, for each pair we know for certain the member who is in the trunk, thus we again have at least two nodes in the trunk core.

Further, we analyse the cases depending on the number of nodes of degree three. We need the following technical Lemma.

Lemma 10. If there is a tree with three nodes of degree three a , b , and c and there are edges (a, b) and (b, c) , then the third neighbour of b is of degree one and for any correct embedding a , b , and c are embedded to the different levels.

Proof. Consider a correct embedding φ . Say $\varphi(b) = \text{level}(i)[1]$. If now $\varphi(a) = \text{level}(i)[2]$, both $\text{level}(i+1)[2]$ and $\text{level}(i-1)[2]$ are occupied by neighbours of a , so no matter where we embed c , say to $\text{level}(i+1)[1]$ there would be only

one spare slot, $level(i + 2)[1]$ in this case, for two neighbours of c . Recall that we have a tree so a and c can't share more than one neighbour.

So the only possible embedding up to the symmetry is

$$\begin{cases} \varphi(b) = level(i)[1] \\ \varphi(a) = level(i - 1)[1] \\ \varphi(c) = level(i + 1)[1] \end{cases} \quad (16)$$

In this case $level(i \pm 1)[2]$ are occupied by neighbours of a and c , so the third neighbour of b cannot have any neighbours except for b since there is no place for them.

Now, we consider the possible cases for the amount of nodes with degree three.

- There are no nodes of degree three. In this case our tree is just a line-graph l and we embed it the following way:

$$\varphi : V(l) \rightarrow Ladder_n \quad (17)$$

$$\varphi(l[i]) = level(i)[1] \quad (18)$$

Remember that right now we allow to choose any n , arbitrary large.

- There is one node of degree three. We can think of it as two line graphs l_1 and l_2 with additional edge $(l_1[i], l_2[1])$ for some i . We embed the tree in the following way:

$$\varphi : V(l_1) \cup V(l_2) \rightarrow Ladder_n \quad (19)$$

$$\varphi(l_1[j]) = level(j)[1] \quad (20)$$

$$\varphi(l_2[j]) = level(i + j - 1)[2] \quad (21)$$

- There are two nodes of degree three. Since $|V(trunkCore(T))| \leq 1$, we conclude that those two nodes are paired, since otherwise they would be single nodes and therefore be in the trunk core by Lemma 5. So, in this case we can present T as two line-graphs l_1 and l_2 with additional edge $(l_1[i], l_2[j])$ for some i and j . We embed T in the following way:

$$\varphi : V(l_1) \cup V(l_2) \rightarrow Ladder_n \quad (22)$$

$$\varphi(l_1[k]) = level(k)[1] \quad (23)$$

$$\varphi(l_2[k]) = level(i + k - j)[2] \quad (24)$$

- There are three nodes of degree three. Since $|V(trunkCore(T))| \leq 1$, we conclude that there is no single node, otherwise, it is in the trunk core and one of the other two is also in the trunk core, contradicting the assumption. So, with our three nodes of degree three, say a , b and c we must have edges (a, b) and (b, c) . By the Lemma 10 none of a, b, c can be embedded into the same level, and the third neighbour of b is of degree one. Denote the line-graphs connected to a with l_a^1 and l_a^2 (they are line-graphs since we have

only three nodes of degree three) and the line-graphs connected to c with l_c^1 and l_c^2 . Denote the third neighbour of b with d . We embed as follows:

$$\varphi(b) = level(2)[1] \quad (25)$$

$$\varphi(d) = level(2)[2] \quad (26)$$

$$\varphi(a) = level(1)[1] \quad (27)$$

$$\varphi(c) = level(3)[1] \quad (28)$$

$$\varphi(l_a^1[i]) = level(1-i)[1] \quad (29)$$

$$\varphi(l_a^2[i]) = level(2-i)[2] \quad (30)$$

$$\varphi(l_c^1[i]) = level(3+i)[1] \quad (31)$$

$$\varphi(l_c^2[i]) = level(2+i)[2] \quad (32)$$

- There are no other cases, since we showed that if there are four nodes with degree three then the size of the trunk core should be bigger than one.

Now, we discuss how to embed a more generic tree with $|V(trunkCore(T))| \geq 2$ into the grid. We call our embedding as $\tilde{\varphi}$.

1. $\tilde{\varphi}(trunkCore(T)[i]) = level(i)[1]$
2. Suppose l is a simple-graph connected to the inner node with number i of the trunk core by its j -th node, so the leg of l is $(trunkCore[i], l[j])$. We embed $l[j]$ to the opposite of $trunkCore[i]$, i.e. $\tilde{\varphi}(l[j]) = level(i)[2]$. We also want to reserve nodes $level(|V(trunkCore(T))|[2]$ and $level(1)[2]$ for exit-graphs, so we say we embed phantom nodes there for algorithm not to use them on Step 3.
3. We now want to embed hands of simple-graphs connected to the inner trunk core nodes. Suppose we have such simple graph l and we've embedded its head to $level(i)[2]$ on Step 2.

If l is zero-handed, it is already embedded on Step 2.

If l is two-handed, denote its hands with h_1 and h_2 and choose the one of embeddings from

$$\left[\begin{array}{l} \left\{ \begin{array}{l} \tilde{\varphi}(h_1[j]) = level(i+j)[2] \\ \tilde{\varphi}(h_2[j]) = level(i-j)[2] \end{array} \right. \\ \left\{ \begin{array}{l} \tilde{\varphi}(h_1[j]) = level(i-j)[2] \\ \tilde{\varphi}(h_2[j]) = level(i+j)[2] \end{array} \right. \end{array} \right. \quad (33)$$

which does not map nodes from $V(h_1) \cup V(h_2)$ to the place nodes were mapped to on step 2.

If l is one-handed, denote its hand with h and consider two cases:

–

$$\left\{ \begin{array}{l} \text{trunkCore}(T)[i + 1] \text{ is an inner node and it is a foot of another} \\ \text{one-handed or zero-handed simple-graph } l_2 \\ \text{with hand (possibly empty) } h_2 \\ m(h[j]) = \text{level}(i - j)[2] \text{ maps some nodes of } h \text{ to the place} \\ \text{where nodes were placed on step 2} \end{array} \right. \quad (34)$$

(35)

In this case we define $\tilde{\varphi}$ for l and l_2 at a time the following way:

$$\left\{ \begin{array}{l} \tilde{\varphi}(h[j]) = \text{level}(i + j)[2] \\ \tilde{\varphi}(h_2[j]) = \text{level}(i + 1 - j)[2] \end{array} \right. \quad (36)$$

– The symmetric case is when

$$\left\{ \begin{array}{l} \text{trunkCore}(T)[i - 1] \text{ is an inner node and it is a foot of another} \\ \text{one-handed or zero-handed simple-graph } l_2 \\ \text{with hand (possibly empty) } h_2 \\ m(h[j]) = \text{level}(i + j)[2] \text{ maps some nodes of } h \text{ to the place} \\ \text{where nodes were placed on step 2} \end{array} \right. \quad (37)$$

(38)

In this case we define $\tilde{\varphi}$ for l and l_2 at a time the following way:

$$\left\{ \begin{array}{l} \tilde{\varphi}(h[j]) = \text{level}(i - j)[2] \\ \tilde{\varphi}(h_2[j]) = \text{level}(i - 1 + j)[2] \end{array} \right. \quad (39)$$

– If the previous two cases don't come true we act pretty much the similar as we did for two-handed simple-graph, namely denote hand of l with h and choose one of the following definitions of $\tilde{\varphi}$ which doesn't map nodes of h to the places already used on step 2:

$$\left[\begin{array}{l} \tilde{\varphi}(h[j]) = \text{level}(i + j)[2] \\ \tilde{\varphi}(h[j]) = \text{level}(i - j)[2] \end{array} \right. \quad (40)$$

4. The last case is to consider an exit-graph.

Denote the end-node of the trunk core, to which the exit-graph is connected by i . i is either $|V(\text{trunkCore}(T))|$ or 1.

– If $i = |V(\text{trunkCore}(T))|$. There are two line-graphs connected to i , say l_1 and l_2 . Note that they can't both be two-handed, since that means we have three nodes of degree three in a row, the middle one is the end node of the trunk core, but then the middle one by Lemma 10 must have the third neighbour of degree one which is not the case since it is a trunk core node and trunk core nodes are all of degree ≥ 2 .

So let's assume that l_1 is one-handed. We embed it with

$$\tilde{\varphi}(l_1[j]) = \text{level}(i + j)[1] \quad (41)$$

If l_2 is one-handed we embed it with

$$\tilde{\varphi}(l_2[j]) = \text{level}(i + j - 1)[2] \quad (42)$$

If l_2 instead has two hands h_1 and h_2 and it connects to i with the node $l_2[j]$. Then we define

$$\tilde{\varphi}(l_2[j]) = \text{level}(i)[2] \quad (43)$$

$$\tilde{\varphi}(h_1[k]) = \text{level}(i + k)[2] \quad (44)$$

$$\tilde{\varphi}(h_2[k]) = \text{level}(i + k)[2] \quad (45)$$

- If $i = 1$ we do everything symmetrically. Remember we don't care if we go out $Ladder_n$ top or bottom borders, if it happens, we can just enlarge our grid to $Ladder_m$ for some large enough m to accommodate the image.

Definition 21. *The definition of φ on the hand(s) of the simple-graph connected to the inner trunk core node is called the orientation of that simple-graph.*

Definition 22. *We say that two inner simple-graphs are neighbours if there are no other simple-graphs connected to the trunk core in between their foots.*

Lemma 11. *The resulting embedding of this strategy exists and it is quasi-correct.*

Before diving into prove let us discuss what does the Lemma give to us. There are three key points about the described quasi-correct embedding.

First of all, we should emphasise that such embedding can be efficiently computed.

Not only that, but it also can be recomputed easily while remaining quasi-correctness when the new vertices come, which is relevant to the online scenario.

And last but not least, recall that each two adjacent nodes are embedded at the distance at most five (see Lemma 7) so we are not worried about serving the same request many times.

Proof. The lemma is obvious for the trees with $|V(\text{trunkCore}(T))| \leq 1$, since we can do a correct embedding.

Denote the resulting embedding with $\tilde{\varphi}$. We know that a correct embedding exists, denote it φ .

- The described embedding of the trunk core meets no constraints, so it always exists.

- Let $top := |V(trunkCore(T))|$.
 The described embedding of the exit-graphs does not have any constraints, so it exists. Let us now focus on the exit-graphs connected to $trunkCore(T)[top]$. For each node u of those exit-graphs it is true that $\tilde{\varphi}(u)$ is on the $level(top)$ or higher. There are no more than three nodes of exit-graphs per level. Simple-graphs connected to the inner trunk core nodes are not allowed to pass through $level(top)$, so since they are connected, no nodes from simple graphs are embedded into levels $\geq top$. There are no nodes of the trunk core higher than top and on $level(top)$ there are only one node from our exit-graphs. The exit graphs connected to the $trunkCore(T)[1]$ are all embedded to the levels ≤ 1 , so they can't interfere with the exit-graphs connected to the $trunkCore(T)[top]$. Thus we conclude that nodes of exit-graphs connected to the $trunkCore(T)[top]$ do not violate quasi-correctness since there are no more than three nodes on their levels. The same for the exit-graphs connected to the $trunkCore(T)[1]$.
- Now to the two-handed inner simple-graphs. The leg of each such simple-graph for any correct embedding must be embedded horizontally, i.e. $\varphi(foot)$ and $\varphi(head)$ must be on the same level. This is since we know that the trunk core image is monotone by Lemma 10 and it can't be if $\varphi(foot)$ and $\varphi(head)$ are on the different levels:
 Say $\varphi(foot) = level(i)[1]$ and $\varphi(head) = level(i+1)[1]$. Then $level(i+1)[2]$ and $level(i+2)[1]$ are occupied by $head$ neighbours since it is of degree three. The $foot$ is also of degree three because it is an inner trunk core node with a simple-graph connected to it. Thus $level(i-1)[1]$ and $level(i)[2]$ are occupied with its neighbours, trunk core nodes. But the node mapped to $level(i)[2]$ can't be the end node of the trunk core since then it is of degree three and the node mapped to $level(i+1)[2]$ is its neighbour thus we obtain a cycle $\varphi^{-1}(\{level(i)[1], level(i+1)[1], level(i+1)[2], level(i)[2]\})$. So there is another trunk core node after it and it is inevitably mapped to $level(i-1)[2]$ violating monotone property of the trunk core embedding.
 Now denote the hands of our two-handed graph with h_1 and h_2 and let's say that $\varphi(foot) = level(i)[1]$, $\varphi(head) = level(i)[2]$ and $\varphi(h_1[1]) = level(i+1)[2]$. Then $\varphi(h_1[2])$ must be $level(i+2)[2]$ since $level(i+1)[1]$ is occupied by the $foot$ trunk core neighbour $next$ (remember $foot$ is inner). If now $next$ is of degree three we obtain a conflict or a cycle, since $next$'s neighbour occupies $level(i+2)[2]$. If not, $next$ is an inner trunk core node and we continue with the $level(i+3)[2]$ for the $h_1[3]$ and $level(i+3)[1]$ for the next trunk core node of $next$. So we do until we ran out of h_1 nodes. We now say that there are no nodes of degree three in

$$\{trunkCore(T)[i+j] \mid j \in [|V(h_1)|]\} \quad (46)$$

since if there is j such that $trunkCore(T)[i+j]$ is of degree three, we obtain a conflict between the third neighbour of $trunkCore(T)[i+j]$ and $h_1[j]$. That means that $\tilde{\varphi}(h_1[j]) = level(i+j)$ will not place a node to the slot already occupied on step 2 of the strategy. The same for h_2 . We call this line of reasoning the *inductive argument*.

So we proved that for each two-handed simple-graph connected to the inner node of a trunk core one of its orientations will not face conflicts with a neighbours of a trunk core nodes of degree three. Or in other words two-handed inner graphs can't violate the existence of the described embedding.

– We've shown that the quasi-correctness can't be violated on the levels $\geq top$ and ≤ 1 . So now we need to proof that it is not violated in between.

To violate the quasi-correctness we need to obtain at least four nodes per level. Since on each level between top and 1 there is a node from the trunk core and there are no nodes from exit-graphs we conclude that there must be at least three nodes of an inner-simple graphs. And note also that they must be from the different simple-graphs since we don't embed more than one node from one inner simple-graph per level. Denote those simple-graphs with a, b, c . Their foots are somehow ordered in the trunk core, say $foot(b)$ is between $foot(a)$ and $foot(c)$. Since simple-graphs hands conflict at some node, we conclude that either hand of a crosses the $head(b)$ or a hand of c crosses the $head(b)$, otherwise a and c just don't share nodes. W.l.o.g. hand of a crosses $head(b)$. But this is only possible when b and a are one-handed graphs with adjacent foots and in this case their hands are oriented contrary and they only have two conflicts: $hand(a)[1]$ is embedded to the same node as $head(b)$ and $hand(b)[1]$ is embedded to the same node as $head(a)$. So c can possibly participate in that conflict only if c is a one-handed graph with a foot adjacent to $foot(b)$. That is because by our strategy two-handed simple-graphs do not cross other simple-graphs heads at all and the one-handed do only if their foots are adjacent. Our goal now is to show that in such setting c can be oriented the other direction to avoid conflict with b .

Note that we have three nodes of degree three and edges $(foot(a), foot(b)), (foot(b), foot(c))$. This is exactly the statement of Lemma 10, so we conclude that we have the following structure up to symmetry:

$$\varphi(foot(b)) = level(i)[1] \quad (47)$$

$$\varphi(foot(a)) = level(i-1)[1] \quad (48)$$

$$\varphi(foot(c)) = level(i+1)[1] \quad (49)$$

$$\varphi(head(b)) = level(i)[2] \quad (50)$$

$$(51)$$

and we know that b in fact consists of one node.

We still have two possibilities for $head(c)$, namely $level(i+1)[2]$ or $level(i+2)[1]$.

If $\varphi(head(c)) = level(i+1)[2]$, then $\varphi(c[2]) = level(i+2)[2]$ and by the inductive argument applied to the $hand(c)$ there are no nodes of degree three in

$$\{trunkCore(T)[i+1+j] \mid j \in [|V(hand(c))|]\} \quad (52)$$

so

$$\tilde{\varphi}(hand(c)[j]) = level(i+1+j)[2] \quad (53)$$

won't place nodes of c to the slots already occupied on step 2 of the strategy. If on the other hand $\varphi(\text{head}(c)) = \text{level}(i+2)[1]$ then $\varphi(\text{trunkCore}(i+2))$ must be $\text{level}(i+1)[2]$. Thus $\text{trunkCore}(i+2)$ can't be the end of the trunk core since then it is of degree three but $\text{level}(i)[2]$ is occupied by $\text{head}(b)$. So we say that $\text{trunkCore}(i+3)$ exists and $\varphi(\text{trunkCore}(i+3)) = \text{level}(i+2)[2]$ and it is also not the end node since it can't be of degree three since $\text{level}(i+2)[1]$ is occupied by the assumption by the $\text{head}(c)$. We now apply the inductive argument obtaining that there are no nodes of degree three in

$$\{\text{trunkCore}(T)[i+2+j] \mid j \in [|V(c)|]\} \quad (54)$$

We also showed that $\text{trunkCore}(T)[i+2]$ is not of degree three, so we state that $\tilde{\varphi}(\text{hand}(c)[j]) = \text{level}(i+1+j)$ won't place nodes of c to the slots already occupied on step 2 of the strategy.

This completes the proof of quasi-correctness of the embedding.

- So the last thing to show is that the described embedding exists for one-handed inner graphs.

Suppose we have a one-handed inner graph l with hand h connected to the i -th node of the trunk core. Suppose also that w.l.o.g. $\varphi(\text{foot}(l)) = \text{level}(i)[1]$. The only constrained case in our strategy is when the following doesn't hold:

$$\left[\begin{array}{l} \left\{ \begin{array}{l} \text{trunkCore}(T)[i+1] \text{ is an inner node and it is a foot} \\ \text{of another one-handed or zero-} \\ \text{handed simple-graph } l_2 \text{ with} \\ \text{hand (possibly empty) } h_2 \end{array} \right. \quad (1) \\ \left. \begin{array}{l} m(h[j]) = \text{level}(i-j)[2] \text{ maps some nodes of } h \text{ to the} \\ \text{place where nodes were placed} \\ \text{on step 2} \end{array} \right. \quad (55) \\ \left\{ \begin{array}{l} \text{trunkCore}(T)[i-1] \text{ is an inner node and it is a foot} \\ \text{of another one-handed or zero-} \\ \text{handed simple-graph } l_2 \text{ with} \\ \text{hand (possibly empty) } h_2 \end{array} \right. \quad (2) \\ \left. \begin{array}{l} m(h[j]) = \text{level}(i+j)[2] \text{ maps some nodes of } h \text{ to the} \\ \text{place where nodes were placed} \\ \text{on step 2} \end{array} \right. \end{array} \right.$$

For the proof by contradiction assume now that both $\tilde{\varphi}(h[j]) = \text{level}(i+j)[2]$ and $\tilde{\varphi}(h[j]) = \text{level}(i-j)[2]$ map the nodes of h to the places already used on step 2. By the inductive argument that means that there exist such $j_1, j_2 \leq |V(h)|$ that $\text{trunkCore}(i+j_1)$ and $\text{trunkCore}(i-j_2)$ are of degree three. But that means that $\varphi(\text{head}(l)) \neq \text{level}(i)[2]$ since in that case by the inductive argument $\varphi(h[j])$ must be either $\text{level}(i-j)[2]$ or $\text{level}(i+j)[2]$ but in the first case we obtain a conflict with a neighbour of $\text{trunkCore}(i+j_1)$ and in the second case we obtain a conflict with $\text{trunkCore}(i-j_2)$. So, $\varphi(\text{head}(l))$ is either $\text{level}(i+1)[1]$ or $\text{level}(i-1)[1]$. Let's consider the case $\text{level}(i+1)[1]$, the second is totally symmetric.

The trunk core nodes adjacent to $foot(l)$ are $trunkCore(T)[i - 1]$ and $trunkCore(T)[i + 1]$ they are mapped by φ to $level(i - 1)[1]$ and $level(i)[2]$. We consider the case where $\varphi(trunkCore(T)[i - 1]) = level(i - 1)[1]$ and $\varphi(trunkCore(T)[i + 1]) = level(i)[2]$ and we show that in this case (1) holds. Symmetrically $\varphi(trunkCore(T)[i - 1]) = level(i)[2]$ and $\varphi(trunkCore(T)[i + 1]) = level(i - 1)[1]$ will lead to (2).

We now prove that $trunkCore(T)[i + 1]$ can't be the end node of the trunk core.

In Lemma 5 we make a path through the support nodes. If $trunkCore(T)[i + 1]$ is not a support node, it can't be the end node of the trunk core since the trunk core connects to support nodes. It is neither a single node, since it has a neighbour $trunkCore(T)[i]$ of degree three. So, since it is in the trunk core, we deduce that it is of degree three and there are nodes a and c of degree three s.t. there is a path $a \rightarrow trunkCore(T)[i + 1] \rightsquigarrow c$. If a is different from $trunkCore(T)[i]$ then we have three consecutive nodes $trunkCore(T)[i]$, $trunkCore(T)[i + 1]$ and a of degree three, so by the Lemma 10 $\varphi(trunkCore(T)[i])$ is on the same side as $\varphi(trunkCore(T)[i + 1])$, which contradicts the assumption of $\varphi(trunkCore(T)[i]) = level(i)[1]$ and $\varphi(trunkCore(T)[i + 1]) = level(i)[2]$. So there is a node c of degree three which is not adjacent to $trunkCore(T)[i + 1]$ and there is a path $trunkCore(T)[i] \rightarrow trunkCore(T)[i + 1] \rightsquigarrow c$. But then either c or its pair (if it is paired) is in the trunk core meaning that the trunk core passes through $trunkCore(T)[i + 1]$ so it is inner.

Thus the $trunkCore(T)[i + 2]$ exists and it has no other options but to be embedded to $level(i + 1)[2]$ since if it is embedded to $level(i - 1)[2]$ it is embedded to the same level as $trunkCore(T)[i - 1]$ and it violates the trunk core monotone property stated by Lemma 10. So we are now able to apply the inductive argument deducing that there are no nodes of degree three in $\{trunkCore(T)[i + 1 + j] \mid j \in [|V(l)|]\}$. Recall that by our proof by contradiction assumption we have that mapping $m : m(h[j]) = level(i + j)[2]$ maps some nodes of h to the place where nodes were placed on step 2 meaning that there is a node of degree three in $\{trunkCore(T)[i + j] \mid j \in [|V(h)|]\}$. But this implies that the node $trunkCore(T)[i + 1]$ is of degree three, it is inner, so we have an inner simple-graph connected to it, moreover this simple graph is a one-handed graph since otherwise we have three nodes of degree three: $trunkCore(T)[i]$, $trunkCore(T)[i + 1]$ and $head$ of that simple-graph, implying by Lemma 10 that $trunkCore(T)[i]$ and $trunkCore(T)[i + 1]$ are mapped to the same side of the grid which as we know is not the case.

So we have that $m : m(h[j]) = level(i + j)[2]$ maps some nodes of h to the place where nodes were placed on step 2 and that there is a one-handed inner simple-graph connected to the $trunkCore(T)[i + 1]$ which is exactly the case (1).

Embedding with Cycles

Definition 12. A maximal cycle C of a graph G is a cycle in G that cannot be enlarged, i.e., there is no other cycle C' in G such that $V(C) \subsetneq V(C')$.

Definition 13. Consider a graph G and a maximal cycle C of G . The whisker W of C is a line graph inside G such that:

- $V(W) \neq \emptyset$, and $V(W) \cap V(C) = \emptyset$.
- There exists only one edge between the cycle and the whisker (w, c) for $w \in V(W)$ and $c \in V(C)$. Such c is called a foot of W . The nodes of W are enumerated starting from w .
- W is maximal, i.e., there is no W' in G such that W' satisfies previous properties and $V(W) \subsetneq V(W')$.



Definition 14. Suppose we have a graph G that can be correctly embedded into $Ladder_n$ by φ and a cycle C in G . Whiskers W_1 and W_2 of C are called adjacent for the embedding φ if

$$\forall i \in [\min(|V(W_1)|, |V(W_2)|)] (\varphi(W_1[i]), \varphi(W_2[i])) \in E(Ladder_n)$$

Statement 3 For any correct embedding of a cycle C into $Ladder_n$ each level of $Ladder_n$ is either occupied with two nodes of C or not occupied at all.

Proof. Suppose the contradictory, and there exists a correct embedding φ of C such that there is only one node of C , say a , on some level i , i.e., $level(i)[1]$. a has two neighbours in C , which we call b and c . W.l.o.g. we say that $\varphi(b) = level(i-1)[1]$ and $\varphi(c) = level(i+1)$. We define $next_{ab}(x)$ for the node $x \in V(C) \setminus \{a\}$ as the next node in C for x in the direction ab . It is easy to see that if $level\langle\varphi(x)\rangle > i$ then $level\langle\varphi(next_{ab}(x))\rangle > i$ since it cannot be less than i and due to the connectivity of the cycle image and it cannot be equal to i since then $next_{ab}(x) = a$ and then $x = c$ but $level\langle\varphi(c)\rangle = i-1$. $level\langle\varphi(b)\rangle = i+1 \rightarrow level\langle\varphi(next_{ab}(b))\rangle > i \rightarrow level\langle\varphi(next_{ab}(next_{ab}(b)))\rangle > i \rightarrow \dots \rightarrow level\langle\varphi(c)\rangle > i$ which is a contradiction.

Lemma 8. Suppose we have a graph G that can be correctly embedded into $Ladder_n$ and there exists a maximal cycle C in G with at least 6 vertices with two neighbouring whiskers W_1 and W_2 of C , i.e., $(foot(W_1), foot(W_2)) \in E(G)$. Then, W_1 and W_2 are adjacent in any correct embedding of G into $Ladder_N$.

Proof. At first, we show that for every correct embedding $foot(W_1)$ and $foot(W_2)$ are embedded to the same level of the grid. Suppose not. So there exists a correct embedding φ of G s.t. $\varphi(foot(W_1)) = level(i)[1]$ and $\varphi(foot(W_2)) =$

$level(i-1)[1]$. By the Statement 3 $level(i)[2]$ and $level(i-1)[2]$ are also occupied with nodes from cycle. So we deduce that $\varphi(W_1[1]) = level(i+1)[1]$ and $\varphi(W_2[1]) = level(i-2)[1]$. But by Statement 3 it means that there are no nodes of C mapped to the levels $i+1$ and $i-2$ and so due to connectivity of the cycle image there are no more nodes of the cycle, but then there are only four nodes in C .

Now we want to show that $W_1[1]$ and $W_2[1]$ are embedded to the same level of the grid for any correct embedding of G . Suppose not. So there exists a correct embedding φ of G s.t.

$$\varphi(\text{foot}(W_1)) = level(i)[1] \quad (56)$$

$$\varphi(\text{foot}(W_2)) = level(i)[2] \quad (57)$$

$$\varphi(W_1[1]) = level(i+1)[1] \quad (58)$$

$$\varphi(W_2[1]) = level(i-1)[2] \quad (59)$$

But this by Statement 3 implies that there are no nodes of C mapped to levels $i+1$ and $i-1$ and thus there are no more nodes of C at all due to the connectivity of the cycle image. Contradiction, since there are at least 6 nodes in C , not 2.

So for every correct mapping φ of G we know that up to symmetry it does the following:

$$\varphi(\text{foot}(W_1)) = level(i)[1] \quad (60)$$

$$\varphi(\text{foot}(W_2)) = level(i)[2] \quad (61)$$

$$\varphi(W_1[1]) = level(i+1)[1] \quad (62)$$

$$\varphi(W_2[1]) = level(i+1)[2] \quad (63)$$

So there is no other option for $W_1[2]$ and $W_2[2]$ but to be embedded to $level(i+2)[1]$ and $level(i+2)[2]$ respectively and so until we reach the end of either W_1 or W_2 . In other words for any correct embedding φ

$$\forall i \in [\min(|V(W_1)|, |V(W_2)|)] (\varphi(W_1[i]), \varphi(W_2[i])) \in E(Ladder_n) \quad (64)$$

Remark 2. Due to Lemma 8 if the cycle is of length ≥ 6 we can forget about an embedding while talking about adjacent whiskers.

Definition 15. Assume we have a graph G and a maximal cycle C of length at least 6. The frame for C is a subgraph of G induced by vertices of C and $\{W_1[i], W_2[i] \mid i \in [\min(|V(W_1)|, |V(W_2)|)]\}$ for each pair of adjacent whiskers W_1 and W_2 . Adding all the edges $\{(W_1[i], W_2[i]) \mid i \in [\min(|V(W_1)|, |V(W_2)|)]\}$ for each pair of adjacent whiskers W_1 and W_2 makes frame completed.

Lemma 12. If we have a cycle of length at least 6 in a graph which is a subgraph of the request graph then its end nodes of the frame are adjacent in the request graph.

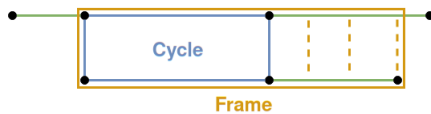


Fig. 9: Cycle, its frame, and edges (dashed) to make the frame completed

Proof. This is because by Lemma 8 they are adjacent for every embedding and in particular for the original embedding of the cycle into the request graph.

Remark 3. Due to Lemma 12 we can “extend” each maximal cycle to the ends of its frame, so, we do not have any adjacent whiskers, i.e., one whisker is embedded fully.

Lemma 13. *Assume we have a graph G which can be embedded into $Ladder_n$ and a maximal cycle C of length at least 6 of G has no adjacent whiskers. Then, there are at most two nodes connected to C (i.e. $(v, c) \in E(G)$, such that $v \in V(G) \setminus V(C) \wedge c \in V(C)$).*

Moreover, these two connecting nodes are not adjacent.

Proof. Consider a correct embedding φ of G into $Ladder_n$. The cycle occupies level from i to j , $i < j$ (it can't make a gap due to the connectivity of the image and by the Lemma 3 it occupies the whole level). So the possible places for v are $level(i - 1) \cup level(j + 1)$.

For the proof by contradiction assume that there are at least three nodes connected to C . Then by the pigeon hole principle there are two of them on the same level, say v_1 and v_2 . There can't be an edge between v_1 and v_2 since then the cycle can be extended by adding v_1 and v_2 and thus is not maximal. But if there is no edge between v_1 and v_2 they form whiskers and those whiskers are adjacent. Contradiction.

Trees and cycles

Definition 23. *By the cycle-tree decomposition of a graph G we mean a set of maximal cycles $\{C_1, \dots, C_n\}$ of G and a set of trees $\{T_1, \dots, T_m\}$ of G such that*

- $\bigcup_{i \in [n]} V(C_i) \cup \bigcup_{i \in [m]} V(T_i) = V(G)$
- $V(C_i) \cap V(C_j) = \emptyset \forall i \neq j$
- $V(T_i) \cap V(T_j) = \emptyset \forall i \neq j$
- $V(T_i) \cap V(C_j) = \emptyset \forall i \in [m], j \in [n]$
- $\forall i \neq j \forall u \in V(T_i) \forall v \in V(T_j) (u, v) \notin E(G)$

Lemma 14. *Assume we have a graph G which can be embedded into $Ladder_n$. Suppose that there is a cycle C and a tree T from cycle-tree decomposition of G , such that C and T are connected by an edge $(c, t) \in E(G)$, where $t \in V(T)$ and $c \in V(C)$. Then, for any correct embedding φ $t \in trunk_\varphi(T)$.*

Proof. By Lemma 3 there is another node of C on the $level\langle\varphi(c)\rangle$. Let's say this level has number i . If $\varphi(t) \in level(i+1)$ then we know that no nodes of T can be embedded to the $level(i)$ (and thus, due to the connectivity of T -s image, below it) so t is the bottom most node and thus it is in the trunk. Symmetrically, it is the top most node of T if $\varphi(t) \in level(i-1)$.

Definition 24. We call a node t from Lemma 14 an end-node of a tree T , and a node c a foot of a T .

Remark 4. If the tree T is connected to a cycle, then $trunkCore(T)$ can be extended to an end-node of T .

We call the path connecting the end-node of the trunk core and an end-node of a tree an extension of the trunk core.

We call a trunk core with two of its possible extensions an extended trunk core.

The exit-graphs are now simple-graphs connected to the end-nodes of an extended trunk core.

Note that the end-nodes of the tree might not exist while the end-nodes of the trunk core are just the end-nodes of the path.

We now define how to embed a tree T from a cycle-tree decomposition.

We include possible foots of a tree with their neighbours in that tree, making them the end-nodes of the trunk core. We then apply strategy C.3 to the obtained tree.

Definition 25. We say that such an embedding of a tree respects the strategy C.3.

C.4 Dynamic algorithm

Now, we talk about how we update the embedding with respect to new requests.

In our strategy of edge processing, if an already known edge is requested we do nothing since the requested nodes are already at the distance at most 12, because by the assumption the enumeration preserves the proximity property (see the strategy plan C.1).

But if we obtain a new edge, our enumeration may no longer maintain the proximity property, so we perform a re-enumeration.

There are two possible cases for the new edge. It may be within the connectivity component or it may connect two different connectivity components. We analyse these cases separately.

We want to maintain the following five invariants:

1. The embedding of any connectivity component is quasi-correct.
2. For each tree in the cycle-tree decomposition the embedding of that tree matches the strategy C.3
3. We do not have maximal cycles of length 4
4. Each maximal cycle does not have adjacent whiskers
5. There are no conflicts with cycle nodes

New edge within one connectivity component Assume we have a connectivity component S with at least one cycle (call it C_S) and a quasi-correct embedding φ' of S preserving all the invariants from C.4.

Assume the new edge connects nodes u and v . Since u and v are already in a one connectivity component we conclude that there is now a cycle C' containing u and v . We consider a maximal cycle C containing C' .

We call a graph S with an edge (u, v) as S_+ .

If C is of length 4, for every two nodes a and b of C $|level\langle\varphi'(a)\rangle - level\langle\varphi'(b)\rangle| \leq 3$ since the distance in S between a and b is at most 3 and φ' preserves connectivity. Thus, since there are no more than 3 nodes per level, we conclude that the difference between numbers of a and b is at most 12, so, the proximity property is maintained and we do nothing.

If C is now of length at least 6 (note that if a cycle can be embedded into $Ladder_n$ its length must be even) we consider its frame F stating that it is in fact a cycle by Lemma 12 and that it has at most two nodes connected to it by Lemma 13.

Lemma 15. *Consider a graph G embedded into $Ladder_n$ with some embedding φ that respects invariants C.4. Consider a frame of a maximal cycle C in G embedded by φ into levels from i to j ($i < j$). Then $\{v \mid v \in V(G), level\langle\varphi(v)\rangle > j\}$ form a connectivity component.*

Proof. By the invariant 4 C.4 and Lemma 13 there is only one node u connected to C with $level\langle\varphi(u)\rangle > j$. So, if some path from a to b ($level\langle\varphi(a)\rangle > j$, $level\langle\varphi(b)\rangle > j$) goes through a node v with $level\langle\varphi(v)\rangle \leq j$ it must pass through u twice, meaning we can replace $a \rightsquigarrow u \rightsquigarrow v \rightsquigarrow u \rightsquigarrow b$ with $a \rightsquigarrow u \rightsquigarrow b$.

We say that a group of nodes is on the same side from a cycle if they are in a one connectivity component when the cycle is removed.

Lemma 16. *If after the removal of F , a connectivity component S_i $i \in \{1, 2\}$ is a line-graph and it connects to F via its end-node then all of its nodes belong to an exit-graph in S .*

Proof. Since by our assumption S has a cycle, we state that each tree has a non-empty extended trunk core and, thus, each node of the component has only four options where to belong. It is either in a cycle, an extended trunk core, an inner-graph, or an exit-graph. So, we now prove that the first three do not happen here:

- The node remains on a cycle when adding a new edge and the nodes from S_i are not on the cycle in S_+ , thus, they were not in S .
- Each node in the extended trunk core is either of degree three or has two edge-disjoint paths to nodes of degree three. Those properties can't disappear when adding a new edge and none of them hold in S_+ for nodes in l thus did not them in S . So the nodes from l were not in the extended trunk core.

- As shown in C.3 for every correct embedding nodes of a simple-graphs always have a node embedded to the same level, namely a node from the trunk core. But S_+ has an embedding with all the nodes from S_i being single on their level, and since every embedding for S_+ induces an embedding for S the same holds for them in S thus none of them is a node of an inner-graph in S .

Lemma 17. *If a connectivity component l left when removing F is a line-graph and it connects to F via its inner node u then all of its nodes belong to an exit-graph in S except possibly u .*

Proof. The proof is almost the same as in 16 but with to adjustments:

- The second bullet does not hold for u
- l has an embedding where a node from l is single on its level or with another node of l and since nodes of l are not the inner trunk core nodes that means that they are not nodes of inner simple-graphs.

By Lemma 15 there are at most two connectivity components left when removing F . Let's call them S_1 and S_2 . We now describe how we embed F , S_1 and S_2 .

So, imagine that we formed F and it has node f of degree three. We first discuss the case of S_1 being a line-graph connected with f with its end-node. By Lemma 16 we deduce that S_1 was a part of an exit-graph in S and thus it was embedded strictly monotonically. In other words, (let's set an enumeration of S_1 such that $(S_1[1], f) \in E(S_+)$) $level\langle\varphi(S_1[1])\rangle < level\langle\varphi(l[2])\rangle < \dots$. We then embed F in the way that f is embedded higher then any other node of F , say to $level(i)[1]$. And we embed $S_1[j]$ to $level(i+j)[1]$. If the levels of nodes of S_1 were decreasing we act the same way but embedding f lower then other nodes of F and embedding S_1 in decreasing order.

What if now S_1 is a line-graph connected to F via its inner node. By the Lemma 17 we know that its hands (call them h_1 and h_2) were exit-graphs and thus were embedded monotonically, assume increasingly numerating from head. Assume head of S_1 was connected to $f \in V(F)$. We then embed F in a way that f is embedded higher then any other node of F say to $level(i)[1]$. We then embed

$$head(S_1) \rightarrow level(i+1)[1] \tag{65}$$

$$h_1[j] \rightarrow level(i+1+j)[1] \tag{66}$$

$$h_2[j] \rightarrow level(i+j)[2] \tag{67}$$

We act symmetrically if the order on h_i was decreasing.

We now want to show that we cannot face incompatibility namely that we have two line-graphs connected (no matter via their inner or end nodes) to F both having increasing (decreasing) order on their hands in S . By Lemma 16 those line-graphs were both whiskers in S , so we denote them W_1 and W_2 . If they both had the same order that means they were in the same tree. To see this

we first prove that there are at most two trees from the cycle-tree decomposition of the component can have exit-graphs. Consider the embedding satisfying invariants C.4. It induces an order on the maximal cycle of the component, since maximal cycles do not intersect and there for can be enumerated from bottom to top for example. If the tree is embedded between two consecutive cycles (say, C_1 and C_2) it must be connected to both of them. This is because they are connected with some path connecting nodes c_1 and c_2 of cycles. We consider such path that contains just two nodes from cycles, it is straightforward to see that such path can be obtained if we have an arbitrary one. This path (except c_1 and c_2) belongs to some tree T in a cycle-tree decomposition. If now some tree different from T (say, T_2) from a cycle-tree decomposition is embedded between cycles, since the component is connected it has two options: either to be connected to T or to one of the cycles. It can't be connected to T by the definition of the cycle-tree decomposition. Neither it can be connected to C_1 or C_2 since by the Lemma 15 if that cycle is removed since T and T_2 are on the one side of that cycle they are in the one connectivity component. But this implies that T_2 is connected to other cycle which is impossible due to the Lemma 15.

The only way for a tree to have exit-graphs is to be embedded below the lowest cycle or above the highest. But exit-graphs in the highest tree are embedded increasingly and the whiskers in the lowest tree are embedded decreasingly.

This means that W_1 and W_2 are in the same tree in S , or in other words, they are both on the same side from C_S . But now this cycle is contained in F (since we only have F, W_1, W_2) and since W_1 and W_2 by Lemma 13 are now on the opposite sides of F they are on the opposite sides to the C_S , which is impossible since then removing C_S leaves W_1 and W_2 in the different connectivity components which was not the case in S .

Lemma 18. *Assume that S_i $i \in \{1, 2\}$ connects to F via a node u , $u \in V(T)$ where T is a tree from the cycle-tree decomposition of S_i and S_i has a node of degree three. Then all the nodes from an extended trunk core of T are embedded monotonically in φ' .*

Proof. All the nodes from a trunk core of T belong to a trunk core in S . This is because T is contained in some tree T_S from a cycle-tree decomposition of S and thus a trunk core of T is contained in a trunk core of T_S since all the support nodes remain support nodes when extending a tree.

So we consider two extensions of T -s trunk core ext_1 and ext_2 . Let's say ext_1 is the one which extends to F . Note that ext_2 was an extension in S and thus we already have that $trunkCore(T) \cup ext_2$ is embedded monotonically by φ' .

The ext_1 was either a part of an extended trunk core of T_S or an exit-graph in T_S . This is because it is obviously couldn't have been a part of a cycle, since nodes on cycle remain on cycle when the new edge is added. Neither could it have been a part of the inner simple-graph since then the end-node of the $trunkCore(T)$ to which ext_1 is connected was an inner node of the trunk core meaning that there was two edge-disjoint paths from it to two support nodes. If those support nodes are in T now the end-node of $trunkCore(T)$ is an inner

trunk core node of T which is nonsense. Otherwise it is not in S_i meaning the path to it goes through ext_1 since it is the only path connecting T and F . But then ext_1 belongs to a trunk core in S . So ext_1 was either a part of an exit-graph or a part of the $trunkCore(T_S)$. In both cases it is embedded monotonically with $trunkCore(T) \cup ext_2$.

So assume we have S_i $i \in \{1, 2\}$ which connects to $f \in V(F)$ via a node u , $u \in V(T)$ where T is a tree from the cycle-tree decomposition of S_i and S_i has a node of degree three. Lemma 18 tells us that T -s extended trunk core was embedded monotonically, say, increasingly starting from u . In this case we embed F the way that f is embedded higher then every other node from F , say, to $level(i)[1]$, and we embed j -th node of an extended trunk core of T to the $level(i + j)[1]$. All the other nodes from S_i we embed the same as they were embedded by φ' relatively to the nodes of the extended trunk core of T .

We now analyze if we obtained a conflict of nodes from S_i and F and if T is embedded respecting strategy C.3.

Consider inner simple-graphs of T in order they are connected to the extended trunk core of T going through the extended trunk core from u . We state that all of them except possibly the first two were inner simple-graphs in S . This is because for a foot of such simple-graph there is a support node before it (one of the foots of first two simple-graphs) and a support node or an end-node after it (because it is inner in S_+). So those simple-graphs can not conflict with nodes of F since they did not pass through the levels of foots of first two simple-graphs. They also respect the strategy C.3.

So we only need to orient first two simple-graphs in the way they don't conflict with nodes of F and they respect the embedding strategy. This can be done, since the strategy can be applied to $T \cup \{f$ and its neighbours}.

Our last goal in analyzing an embedding of a component with a node of degree three which starts with a tree is to show that we can't face incompatibility. If say, S_1 is starting with a tree T with increasing order on its extended trunk core in S then the order on S_2 (if one exists) is decreasing (meaning that S_2 has a decreasing order in S on an extended trunk core of a tree it starts with or a decreasing order in S on its hands if S_2 is just a line-graph). Let's say S_1 and S_2 are connected to F with nodes $c_1 \in V(S_1)$ and $c_2 \in V(S_2)$ respectively.

First assume we have S_1 starting with a tree T_1 and S_2 starting with a tree T_2 . Both S_1 and S_2 have a node of degree three. For T_i take a node u_i which is a foot of T_i other then the one in F if such exists or the end of the extended trunk core other then the one connected to F . Since S_i has a node of degree three, one of those must exist. A shortest path connecting u_1 and u_2 contain both extended trunk cores of T_1 and T_2 . Since no path in $\varphi'(S)$ is self-crossing the path $u_1 - u_2$ must be monotone by Lemma 9 since no nodes of the path can be embedded to the same level with u_1 and u_2 (the neighbour of u_i is either a cycle node or an exit-graph node). Thus paths $u_1 - c_1$ and $c_2 - u_2$ have the same order in $\varphi'(S)$ so the extended trunk cores of T_1 and T_2 which are contained in $c_1 - u_1$ and $c_2 - u_2$ have opposite order.

Now consider the case when S_1 has a node of degree three and starts with a tree T and S_2 is just a line-graph. Let's say that S_1 connects to $f_1 \in V(F)$ and S_2 connects to $f_2 \in V(F)$. For T take a node u_1 which is a foot of T other than the one in F if such exists or the end of the extended trunk core other than the one connected to F . Since S_1 has a node of degree three, one of those must exist. And let's say we have an increasing order on the hands of S_2 . Consider the top-most node of a hand of S_2 in $\varphi'(S)$, let's call it u_2 . Note that u_2 is the top-most node among all S and can share level only with nodes from exit-graphs. Now consider the shortest path $u_1 - u_2$. It contains an extended trunk core of T . Our goal now is to prove that this path is monotone, that would imply as in the previous case that T -s extended trunk core and S_2 -s hands have different order. To see that it is monotone recall that φ' produces no self-crossing paths and thus, if the path is not monotone, by Lemma 9 we either have u_1 sharing level with some other node from path which is impossible since it is either a cycle node or a trunk-core end-node. Or u_2 shares level with some other node from path which is also impossible since u_2 is the top-most node with only nodes from other exit-graphs hands possibly being on its level.

So we discussed what to do if S_i connects to the F with a node from tree from its cycle-tree decomposition. The last case is when it connects to F with a node of a cycle from its cycle-tree decomposition.

So assume S_1 connects to a node $f_1 \in V(F)$ with a cycle C -s node, say u_1 . And let's assume was the top-most node in C the case when it is the bottom-most is totally symmetric. We embed F the way f_1 becomes a bottom-most node we embed C the way u_1 is the top most node and it is under f_1 . We embed the rest of S_1 relatively to C as it was embedded by φ' .

Nothing changed in S_1 , so the invariants maintain for it. Therefore our only goal is to show that we don't face incompatibility with S_2 .

If S_2 starts with a cycle C_2 which connects to F via a node u_2 then we conclude that u_2 was the bottom-most node of C_2 since there is a path in which connects u_1 and u_2 without passing through other nodes of C_1 and C_2 .

If S_2 starts with a tree we act similarly as we did in previous cases take such node u_2 that the path $u_1 - u_2$ contains tree-s trunk core/extended trunk core/hand of an exit graph and we show that by Lemma 9 path $u_1 - u_2$ should be monotone, since no nodes of it can be embedded to the same level as u_2 for the same reasons as before and neither can they be embedded to the same level with u_1 since there is a cycle node embedded to the same level as u_1 . So if, say, u_1 was the top-most node of C in $\varphi'(S)$ then the path is increasing and thus the trunk core/extended trunk core/hand of an exit graph is also increasing in $\varphi'(S)$ which is consistent to the fact that it connects to the top-most node of F .

All the actions described above assume that there was a cycle in S already. If there wasn't we act as described below.

The new edge (u, v) is in one connectivity component so there is a maximal cycle containing u and v . Let's take a frame F of that cycle. There are possibly

two connectivity components left when removing F from S_+ , denote them S_1 and S_2 . Let's say that S_1 connects to $f_1 \in V(F)$ and S_2 connects to $f_2 \in V(F)$. Moreover we know that S_1 and S_2 are trees.

We embed F the way that f_1 is the top-most node and f_2 is the bottom-most node. We then embed S_1 and S_2 the way they match strategy C.3 orienting the trunk not to conflict with cycle nodes.

C.5 Cost of the algorithm

We now want to analyze the cost of actions performed when serving a new edge within one component. Note that since the resulting embedding is quasi-correct the cost of serving the request is $O(1)$.

To make an amortized analysis we introduce the concept of *scenarios*. The scenario is a reason for node to move in the embedding and thus change its number. Each scenario has two main properties: the number of times it can happen to a certain node (denoted with SC_N) and a cost payed for that node movement in this scenario (denoted with SC_C). So the total cost payed for node movements in this terms is bounded with

$$2n \cdot \sum_{SC \in \text{SCENARIOS}} SC_N \cdot SC_C \quad (68)$$

Where the first factor $2n$ arises due to the fact that SC_N and SC_C are defined for one particular node, but we want the total cost.

We propose that we only need to focus on the relative order changes.

Lemma 19. *If we have two enumerations h_1 and h_2 of graph G then the cost of obtaining h_2 from h_1 via swaps is no more than*

$$|\{(u, v) \mid u, v \in V(G), h_1(u) < h_1(v) \wedge h_2(u) > h_2(v)\}| \quad (69)$$

Proof. We can order nodes of G by h_2 . h_1 can then be viewed as a permutation so the statement of the Lemma can be reformulated as "the swap distance between a permutation p and an identity permutation is less or equal the number of inversions in p ".

We proof this by induction. The induction would be among the number of elements in permutations and among the number of inversions in permutations.

The induction base is 0 inversions for each number of elements which is trivial. We also notice that if there is just one element in the permutation then this permutation can't have inversions.

We now assume that our permutation p has n elements and k inversions, with $k > 0$ and $n > 1$.

If now $p[1] = 1$ then the distance between p and *identity* _{n} is the distance between p' and *identity* _{$n-1$} where $p'[i] = p[i+1] - 1$. And since p' has the same number of inversions as p then by the inductive assumption it is less or equal to k which is what we desire.

If $p[1] = i$, $i \neq 1$ then we first spend $i - 1$ swaps to bring i to the position 1 reducing the number of inversions by $i - 1$. And then apply the same idea with p' obtaining that the distance from p' to an $identity_{n-1}$ is $k - (i - 1)$ and thus we provided the series of swaps to obtain an $identity_n$ from p with $\leq k$.

Lemma 20. *Suppose S is a connectivity component with a cycle embedded into $Ladder_n$ via φ' which respects invariants C.4. Suppose we have a new edge in the connectivity component S . We denote S with a new edge by S_+ . Let's call the frame of a maximal cycle containing the ends of the new edge F . The connectivity components left when removing F from S_+ are S_1 and S_2 . Suppose that our algorithm embeds S_1 above F .*

1. *If there is a cycle in F that was present in S then all the nodes from S_1 were above that cycle in $\varphi'(S)$.*
2. *If there is a cycle in F that was present in S then there is a node of cycle that is top-most for both new and old embeddings.*
3. *For each node $u \in V(S_1)$ and for each node $v \in V(S_2)$ $level\langle\varphi'(u)\rangle > level\langle\varphi'(v)\rangle$ except possibly the first two inner simple-graph nodes of T_i if S_i connects to F with a tree T_i from a cycle-tree decomposition of S_i .*

Proof. Assume S_1 connects to F via edge (f_1, c_1) , $f_1 \in V(F)$, $c_1 \in V(S_1)$ and S_2 connects to F via edge (f_2, c_2) , $f_2 \in V(F)$, $c_2 \in V(S_2)$

1. We want to prove the following fact: consider component A is connected to cycle C with edge (a, c) , $a \in V(A)$ $c \in V(C)$ and it is embedded above C by φ_1 and below C by φ_2 . Then for each path in A starting from a which is monotone for both φ_1 and φ_2 it has changed its orientation i.e. if it was increasing it is now decreasing and vice versa. To see this note that a was a top-most node and became the bottom-most node of the path or vice versa. For each type of S_1 our new edge processing strategy maintained an orientation of some monotone path in S_1 which can be extended remaining monotone to the node connected to the cycle. Thus by the fact above it must remain on the same side of the cycle.
2. Denote the cycle in statement by C . Denote by A the component that was above C in $\varphi'(S)$ which contains S_1 . Say it is connected to C via edge (a, c) $a \in V(A)$ $c \in V(C)$. Then by item 1 node c is top-most in both embeddings of S and S_+ since there is a monotonically increasing path starting from a which does not pass through nodes of C .
3. If S_1 and S_2 are both line-graphs from the proof of compatibility for line-graphs we know that they must be in the different trees in a cycle-tree decomposition of S meaning they are separated by cycle and thus their nodes don't share levels. Moreover nodes from S_1 were in the top-most tree and nodes from S_2 were in the bottom-most tree, so indeed every node from S_1 was embedded higher than every node of S_2 .

For all the other cases on S_1 and S_2 in the proof of their compatibility we build a monotone path which passes through c_2, f_2, f_1, c_1 . This path is monotonically increasing levels in $\varphi'(S)$ if S_1 is embedded above F by our

algorithm. So $level\langle\varphi'(c_2)\rangle < level\langle\varphi'(c_1)\rangle$ so our goal now is to analyze what nodes from S_1 can go under $level\langle\varphi'(c_1)\rangle$ (the analysis for S_2 is symmetric). If S_1 connects to F with a cycle or S_1 is a line-graph no nodes can go under c_1 .

To analyze the last case on S_1 we need the following fact: if S_2 exists then $V(F)$ contains a node of degree three in S . This is because $V(F)$ contains two nodes of degree three in S_+ namely the foots of opposite (non adjacent) whiskers and those nodes are not adjacent since the cycle is of length at least 6. So one of them must have been of degree three before the new edge.

So assume now S_1 connects to F with a tree and contains a node of degree three. From the analysis of compatibility for such S_1 we know that c_1 is either an extended trunk core node or an exit-graph node in S . If it is an exit-graph node, then S_2 does not exist since if it does there is a node of degree three in $V(F)$ in S and thus there are two disjoint paths from c_1 to nodes of degree three which can't be for an exit-graph node.

If now c_1 is in an extended trunk core node of a tree T . If F contains a cycle that was present in S then nodes from S_1 and S_2 were separated by this cycle then by item 1 all the nodes from S_1 were above that cycle and all the nodes from S_2 were below so the proposal holds. All the nodes from an extended trunk core of T were above c_1 in S so the only possible nodes are the nodes to go below c_1 are nodes from inner simple-graphs of T . And the inner simple-graphs starting from the third one can't cross the first two graphs heads so they can't cross c_1 as well.

We are now ready to analyze the cost paid by each node when a new edge in the connectivity component appears.

Scenario 1 (Inner simple-graph reorienting) *The node falls into this scenario if it is a part of an inner simple-graph which is being reoriented.*

Lemma 21. *The “Inner simple-graph reorienting” scenario costs $O(n)$ for a node and happens only once for a given node.*

Proof. Node can't pay more than $2n$ so $O(n)$ bound is trivial.

The inner simple-graph has only two possible orientations. We change its orientation only if the current one violates the invariants C.4 thus we will never get back to it.

Scenario 2 (First time on a cycle) *The node falls into this scenario if it was not on a cycle before the new edge and after the new edge it is.*

Lemma 22. *The “First time on a cycle” scenario happens at most once for each node and costs $O(n)$.*

Proof. Trivial.

Scenario 3 (Cycle in the frame) *The node falls in this scenario if it is in the cycle which is a part of F and is present in S .*

Lemma 23. *The “Cycle in the frame” scenario happens $O(n)$ times for each node and costs $O(1)$.*

Proof. Since we have only $O(n)$ edges the $O(n)$ upper bound is trivial.

If the cycle is embedded respecting invariants C.4 with a node specified to be the top most (which is the case due to the lemma 20) then we have only two four possibilities for a cycle to be embedded and for each to of them one can be transformed to another making each node changing the relative order only with $O(1)$ nodes from cycle which by Lemma 19 gives us $O(1)$ cost per node.

Note also that by Lemma 20 nodes from cycle do not change relative order with nodes from S_1 or S_2 and thus the cost of their inner relative change is the only cost they need to pay.

So the nodes of F fall into either *First time on a cycle* scenario or to the cycle in the frame scenario.

As for the S_i nodes we need to consider 2 cases. The nodes are either a part of first two inner simple-graphs when S_i starts from a tree or not.

If yes, and those node change their relative order to the nodes of S_i this means that the reorientation of simple-graphs had been performed thus they fall into *Inner simple-graph reorienting* scenario. If they did not change relative order to S_i this means they didn’t change the order with S_j , $j \in \{1, 2\} \setminus \{i\}$ neither with F so they pay nothing.

As for the nodes from S_i that are not the part of first two inner simple-graphs by the Lemma 20 they didn’t change the relative order with S_j , by the embedding strategy they didn’t change the relative order with the nodes from S_i (// consider mirroring? //) accept possibly first two inner-simple graphs and also by the Lemma 20 they didn’t change the relative order with the cycle contained in F . As for the nodes of F which were not on the cycle we say that they pay for all the relative order changes.

New edge between two components We now define and analyse the behaviour of the algorithm when the edge between two connectivity components is revealed. The strategy would be to bring the larger component towards the smaller one.

Scenario 4 (Connectivity component movement) . *The node falls in this scenario if its component is a smaller of two between which the new edge is revealed.*

Lemma 24. *The “Connectivity component movement” scenario for a node happens $O(\log n)$ times and cost $O(n)$.*

Proof. The size of the component is at least doubled.

We now dive into the case analysis.

We first want to distinguish to cases: either the bigger component has a tree with a non-empty trunk core or not.

If not that means that there are at most two nodes of degree three. If the new edge increases the number of degree three nodes from 0 to one, from 1 to 2 or from 2 to three we say that it is an individual scenario and we can allow the total reconfiguration according to the C.3.

Scenario 5 (New degree three node.) *The node falls into this scenario if the new edge increases the number of the nodes of degree three in the component from 0 to 1, from 1 to 2 or from 2 to 3.*

Lemma 25. *The “New degree three node” scenario can happen $O(1)$ times and costs $O(n)$.*

Proof. Trivial.

We now assume that there is a tree in the bigger component with a non-empty trunk core or a cycle with an inner edge. The smaller component can then connect to:

1. The inner node of the trunk core.
2. The inner simple-graph node.
3. The cycle node.
4. The exit-graph node.

In the following analysis the new scenario appear

Scenario 6 (No more an exit-graph.) *The node falls into this scenario if it is no longer a part of an exit-graph.*

Lemma 26. *The “No more an exit-graph” scenario happens at most once and costs $O(n)$.*

Proof. The closest node of degree three to the exit-graph node has only one path to another degree three node.

Let us discuss all the possible cases.

1. If the smaller connectivity component connects to the inner trunk core node then by Lemma 6 it must be a line-graphs and there for to maintain the quasi-correctness of the embedding we only to choose its orientation and reorient its trunk core neighbours. The scenarios engaged here are the *connectivity component movement* and *inner simple-graph reorienting*.
2. For the inner simple-graph node the analysis is basically the same as in the previous case.
3. Cases:
 - Only cycle
 - Cycle and a line-graph
 - Cycle and a tree with a degree three node
4. Cases:
 - Smaller component does not make new nodes of degree three
 - It does. Then we reorient the exit-graphs as they are no longer exit-graphs. This is the no more an exit graph scenario.

D Embedding a general demand graph on a line graph

Here we propose an online algorithm for a general demand graph G assuming having an oracle algorithm for solving the Bandwidth problem. Given a long enough request sequence, namely $|\sigma| = \Omega(|E(G)| \cdot |V(G)|^2)$ the proposed algorithm has an $O(\lambda \cdot \text{Bandwidth}(G))$ competitive ratio compared with an optimal offline algorithm. But there is more to it. We point out that this algorithm is robust in a sense that its maximal cost for serving a request exceeds the maximal cost of processing the request (reconfigure + serve) of any online algorithm by at most the factor of λ . Moreover, this algorithm pays at most $O(|E(G)| \cdot |V(G)|^2)$ for reconfiguration in total.

Theorem 5. *Suppose we are given a graph G and an algorithm B , that for any subgraph S of G outputs an embedding $c \in C_{S \rightarrow L_n}$ with the bandwidth less than or equal to $\lambda \cdot \text{Bandwidth}(G)$ for some λ . Then, for any sequence of requests σ with demand graph G there is an algorithm that serves σ with total cost $O(|E(G)| \cdot |V(G)|^2 + \lambda \cdot \text{Bandwidth}(G) \cdot |\sigma|)$. If the number of requests is $\Omega(|E(G)| \cdot |V(G)|^2)$ each request has $O(\lambda \cdot \text{Bandwidth}(G))$ amortized cost.*

Proof. Assume we have processed i requests so far. We get a demand graph built on edges $E_i = \{\sigma_0, \dots, \sigma_i\}$. It induces a subgraph S_i of G . We want to maintain the invariant that each S_i is embedded via $c_i \in C_{S_i \rightarrow L}$ such that bandwidth of h is no greater than $\lambda \cdot \text{Bandwidth}(G)$. Suppose now the embedding c_{i-1} of S_{i-1} respects the invariant and we get a new request σ_i . σ_i is an edge in G , say (u, v) . We have two possibilities: either σ_i is already in S_{i-1} or not. If $(u, v) \in E(S_{i-1})$ then $S_{i-1} = S_i$ and since c_{i-1} respects the invariant we know that $|c_{i-1}(u) - c_{i-1}(v)| \leq \lambda \cdot \text{Bandwidth}(G)$ and hence we take $c_i = c_{i-1}$. If on the opposite $(u, v) \notin E(S_{i-1})$ we take $c_i = B(S_i)$, as an embedding of S_i to the line, and reconfigure the network from scratch.

Now, we analyse the cost. We perform adjustments only for a new revealed edge and, thus, there would be no more than $|E(G)|$ reconfigurations. For each reconfiguration we make at most $|V(G)|^2$ migrations, meaning that the total cost of reconfigurations is at most $|E(G)| \cdot |V(G)|^2$. Since we serve the request after performing a reconfiguration and each configuration has a bandwidth of at most $\lambda \cdot \text{Bandwidth}(G)$ we state that we pay no more than $\lambda \cdot \text{Bandwidth}(G) \cdot |\sigma|$ to serve all the requests.

We now explain how to construct an expensive request sequence, when given an online algorithm ON to obtain a $\text{Bandwidth}(G)$ request processing cost lower bound.

Lemma 3. *Given a demand graph G . For each online algorithm ON there is a request sequence σ_{ON} such that ON serves each request from σ_{ON} for a cost of at least $\text{Bandwidth}(G)$.*

Proof. Consider the resulting numeration φ of $V(G)$ done by ON after serving $r \geq 0$ requests. By the definition of bandwidth, there are such $u, v \in V(G)$ that $|\varphi(u) - \varphi(v)| \geq \text{Bandwidth}(G)$. So, we let the next request $\sigma_{ON}[r+1]$ be (u, v) making ON pay at least $\text{Bandwidth}(G)$ for that request.