Tight Bounds for Online Graph Partitioning

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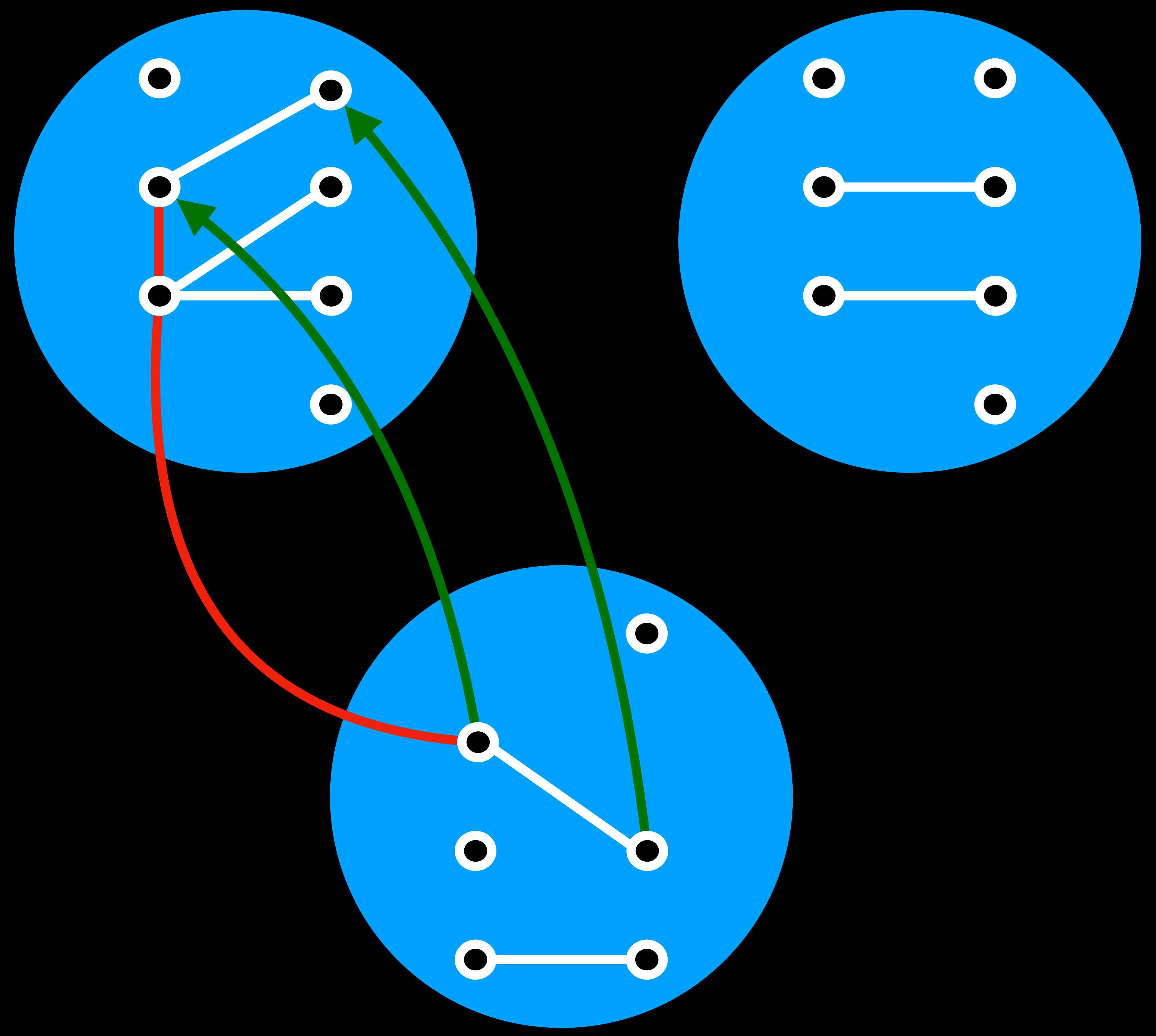
The Problem

Online Graph Partitioning

Problem Definition

- Initially there is an empty graph with n vertices
- Vertices are assigned to ℓ servers
- Each server has capacity for $(1+\varepsilon)\cdot k$ vertices, where $k = n/\ell$
- Adversary inserts edges into the graph
- Vertices of connected components *must* be assigned to the same server
- Algorithm can re-assign components

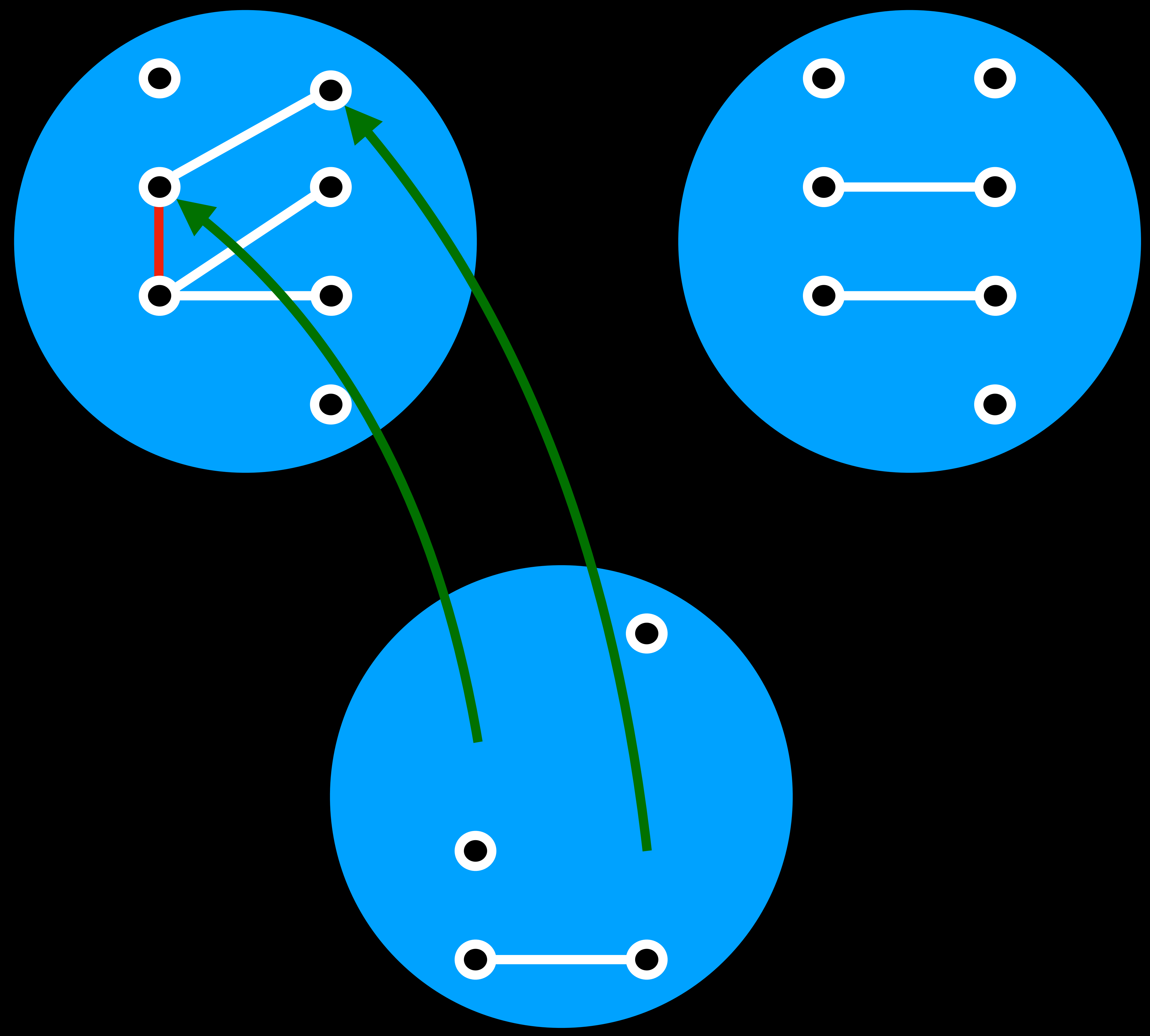


Online Graph Partitioning

Problem Definition

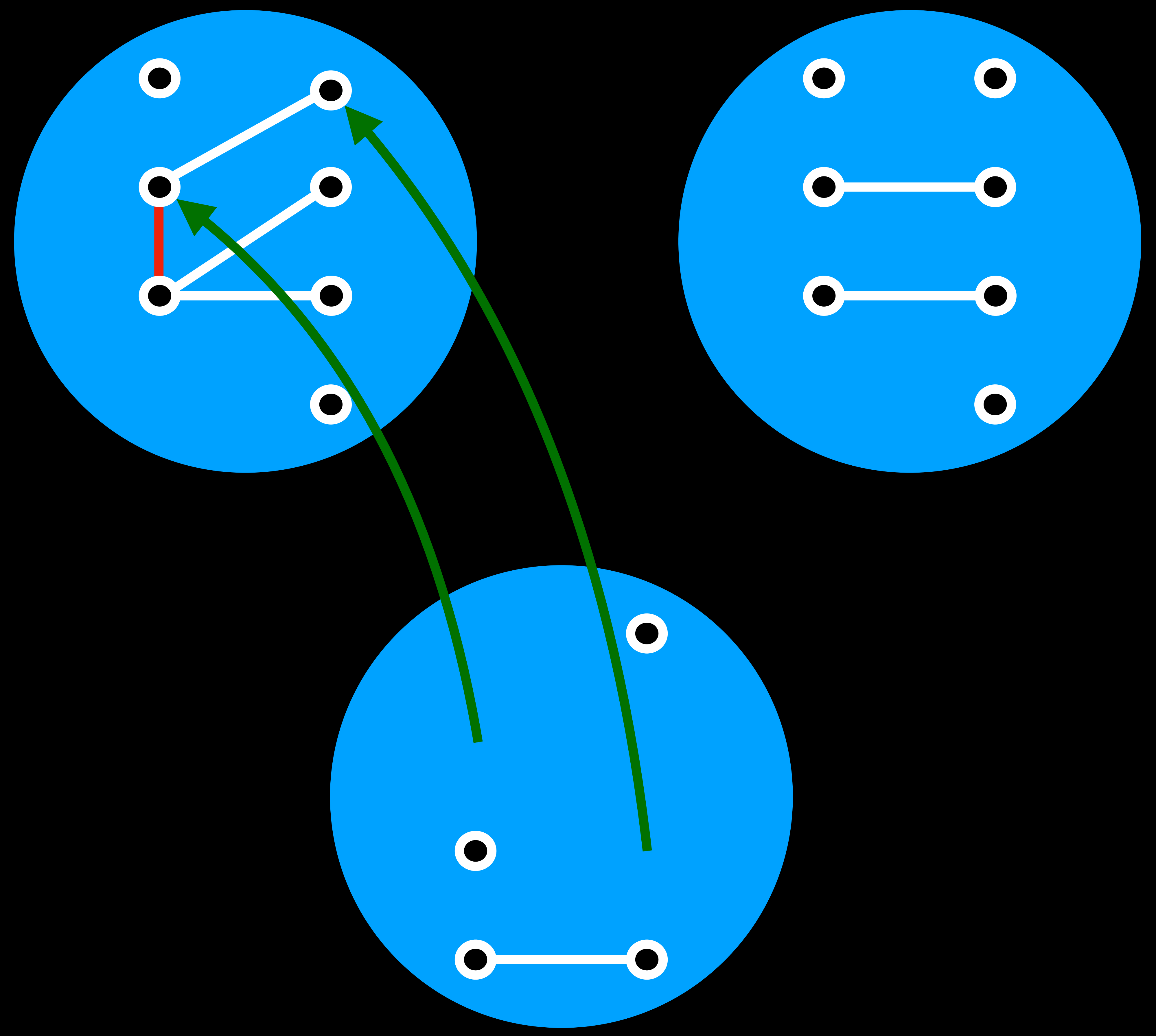
- If ALG moves a connected component with s vertices, it has to pay cost s
- OPT knows all edge insertions in advance and can move to final configuration at minimum cost
- *Goal:* Minimize the (strict) competitive ratio

$$\frac{\text{cost}(\mathrm{ALG})}{\text{cost}(\mathrm{OPT})}$$



Online Graph Partitioning

- Introduced by Henzinger et al. (SIGMETRICS'19)
- Applications:
  - Resource allocation in the cloud
    ➡ servers = datacenters
    ➡ vertices = workloads
    (e.g., communicating virtual machines)
  - Implementing distributed union—find data structures



Our Results

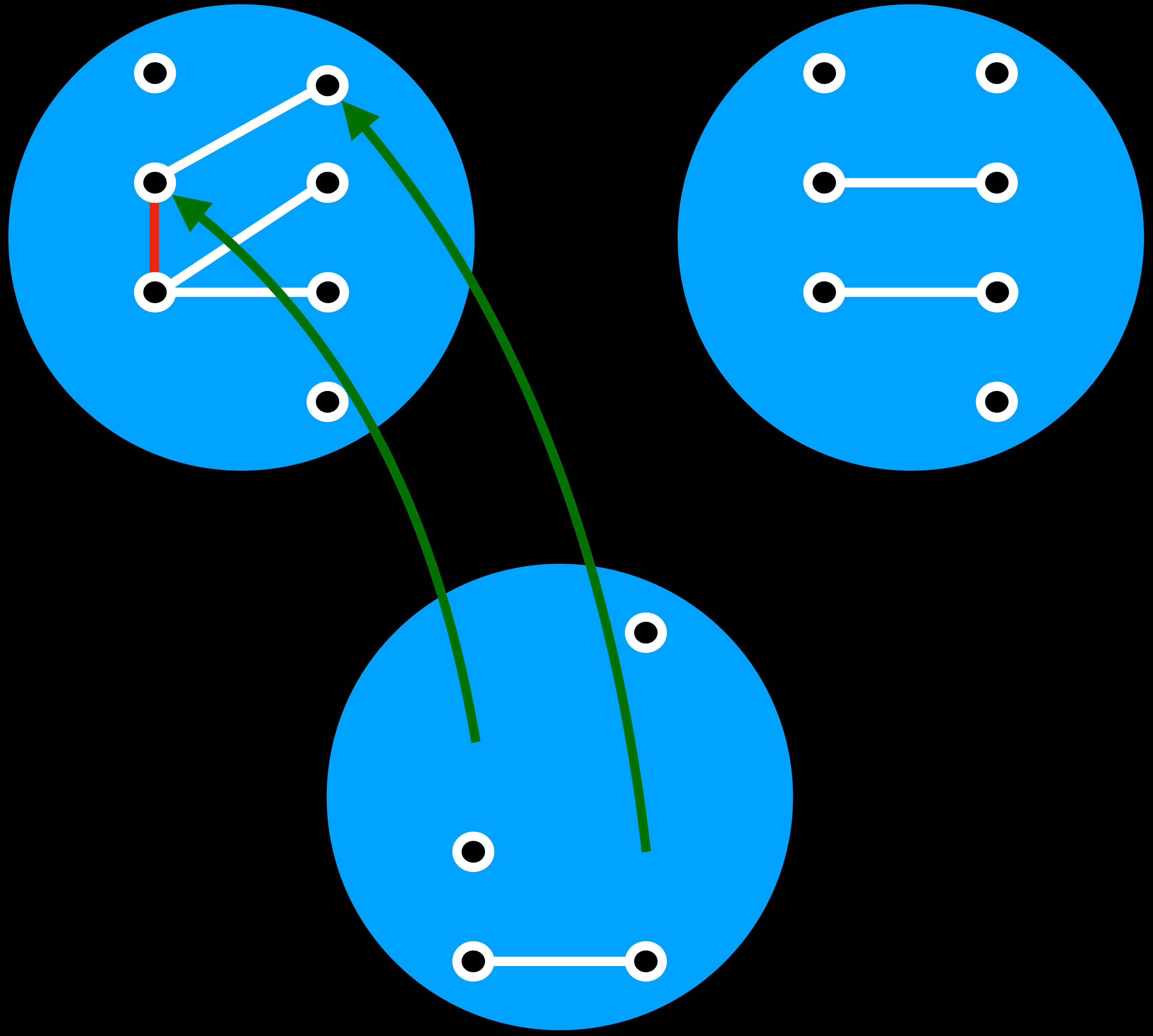
Tight Randomized Algorithm

n vertices
ℓ servers
server capacity = $(1+\varepsilon)k$,
where $k = n/\ell$

- We obtain a randomized algorithm with competitive ratio $O(\log \ell + \log k)$
- We derive a matching lower bound of $\Omega(\log \ell + \log k)$

➡ Our bounds are asymptotically tight

- Exponentially better than the $O(\ell \cdot \log k \cdot \log \ell)$-competitive algorithm in Henzinger et al. (SIGMETRICS'19)



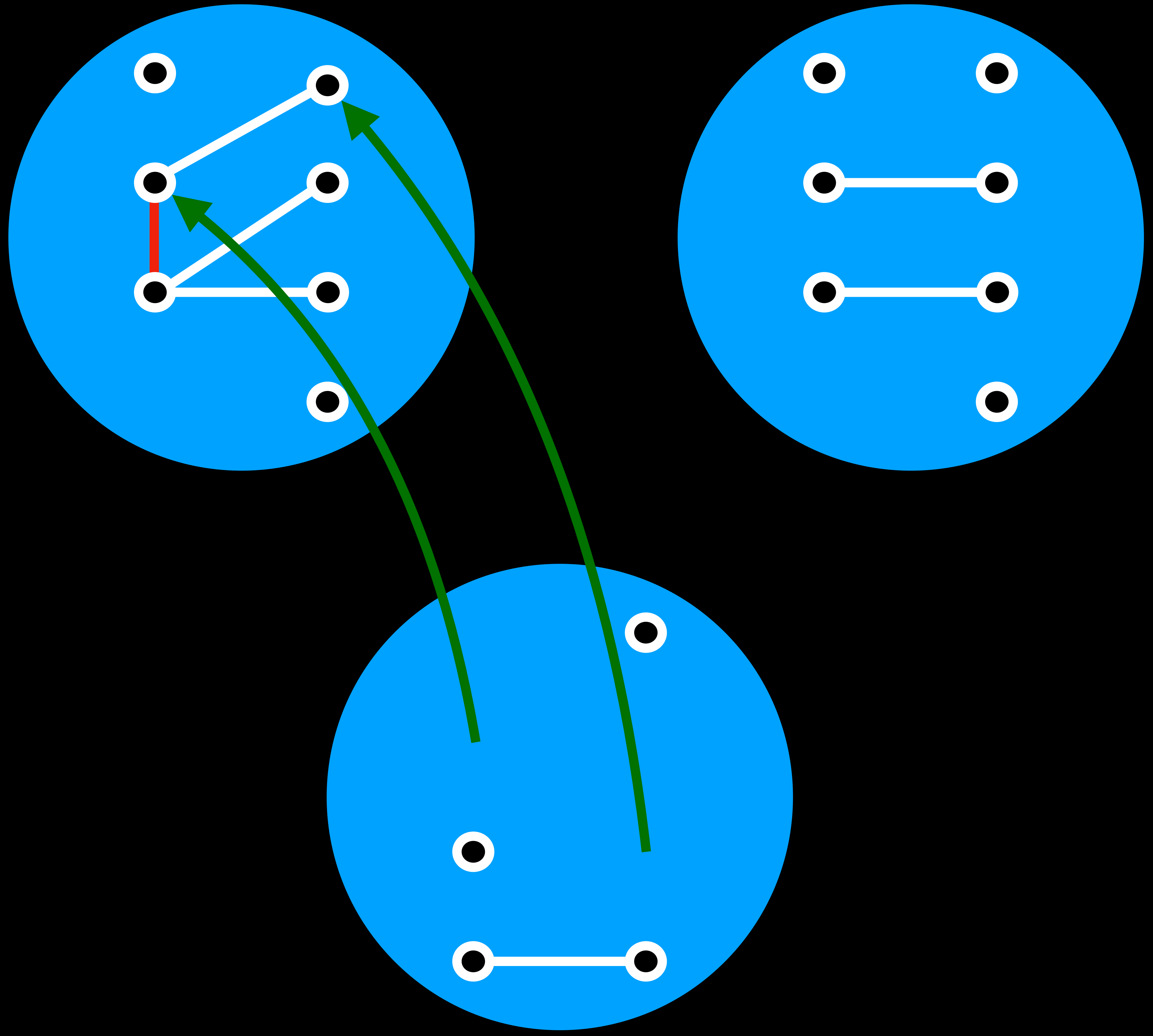
Tight Deterministic Algorithm

n vertices
ℓ servers
server capacity = $(1+\varepsilon)k$,
where $k = n/\ell$

- Deterministic algorithm with competitive ratio $O(\ell \cdot \log k)$
- And matching lower bound of $\Omega(\ell \cdot \log k)$

➡ The bounds are asymptotically tight

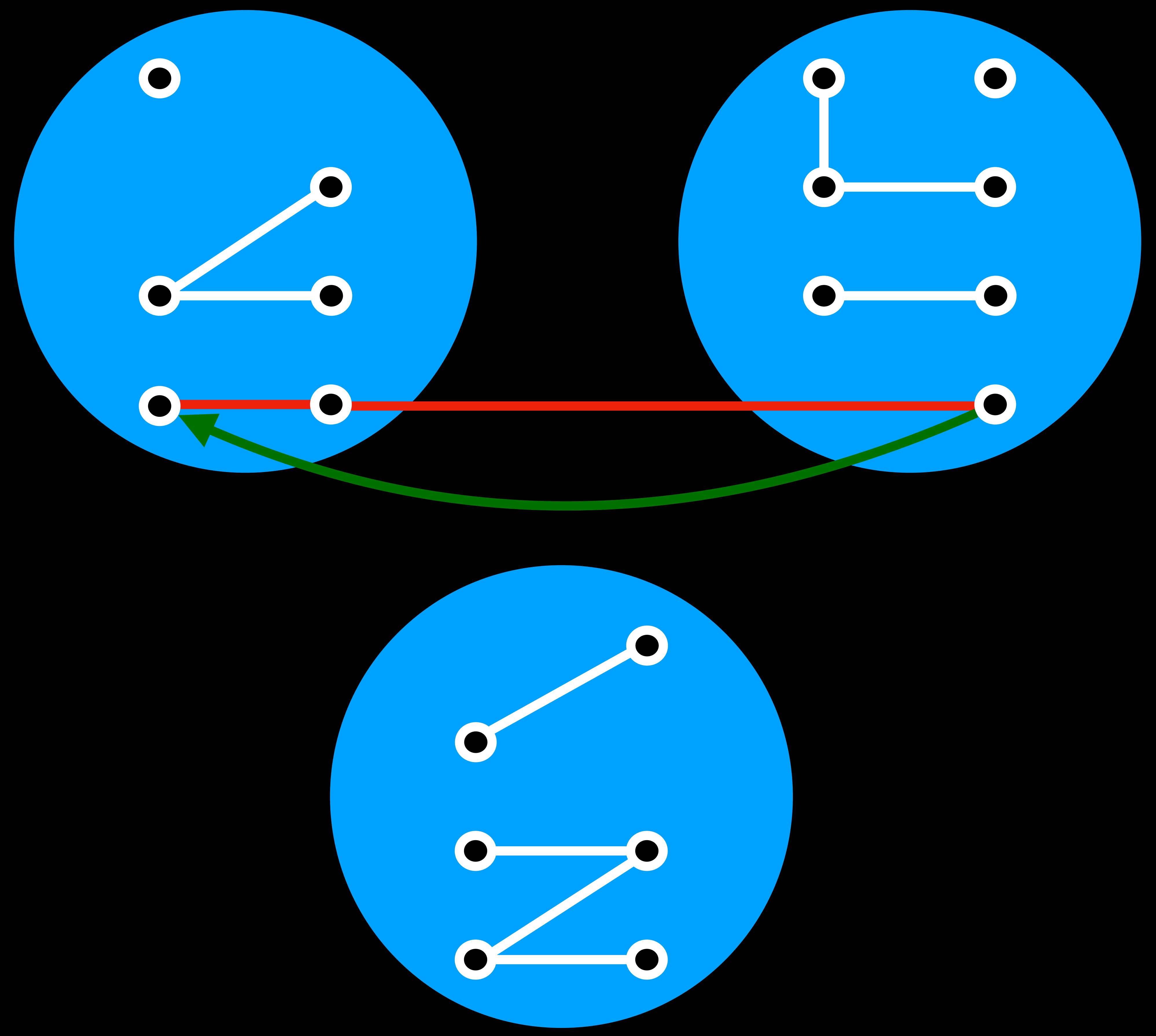
- If $\varepsilon > 1$ (i.e., servers can store $> 2k$ vertices), we get a (tight) competitive ratio of $O(\log k)$



Technical Overview

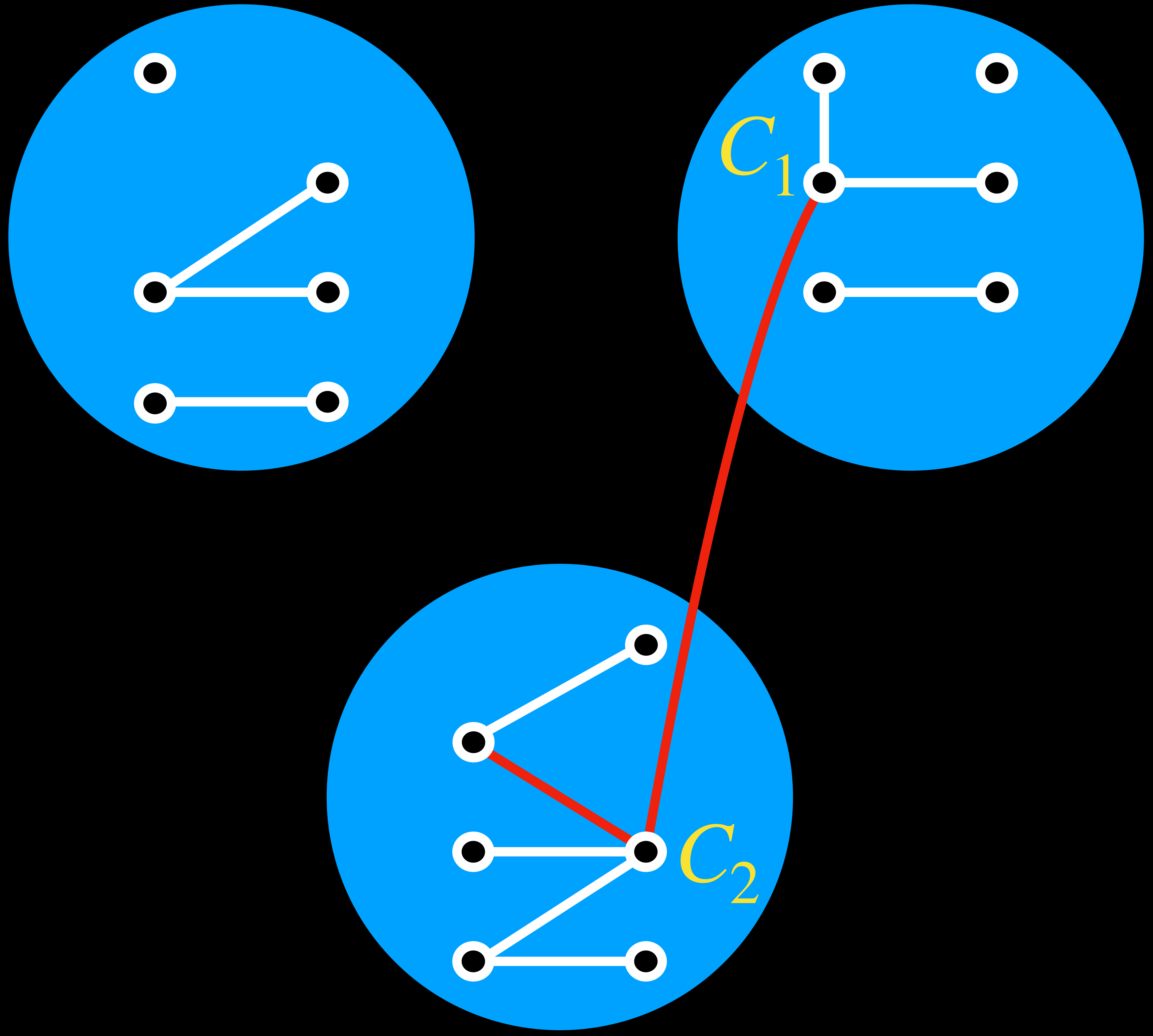
General Problematic

- Reminder:
  - Adversary inserts edges
  - Vertices of connected components *must* be assigned to the same server
  - Goal: minimize the number of vertex moves
- What should we do when an edge is inserted?
- If both components are on the server:
  do nothing
- If components are on different servers:
  need to move vertices



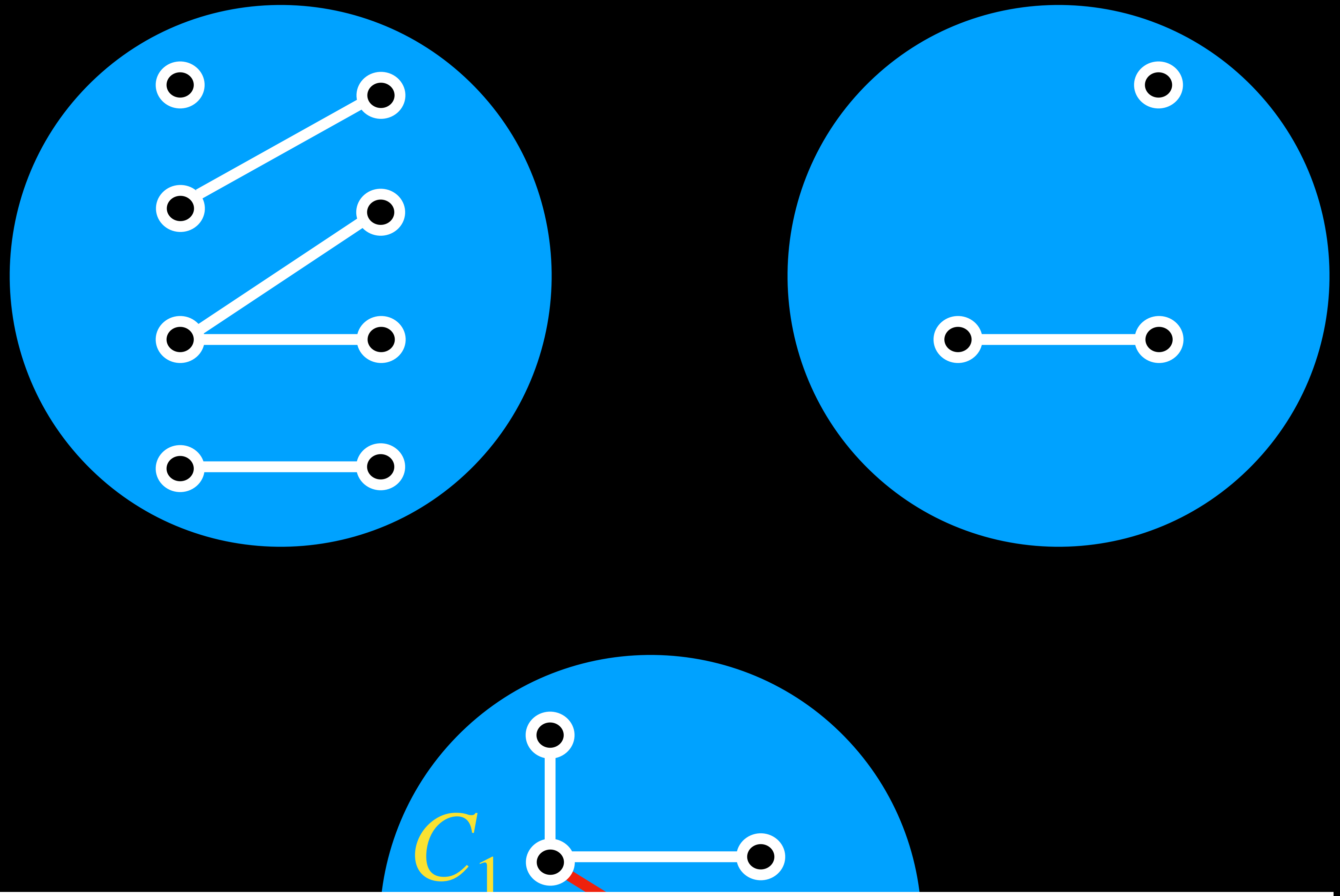
If Components Are on Different Servers

- Connecting two components C_1 and C_2 might trigger a cascade:
  - Not enough server capacity to move C_1 to server of C_2 (and vice versa)
  - We have to move other components as well
- Which components shall we move?



Which Components Shall We Move?

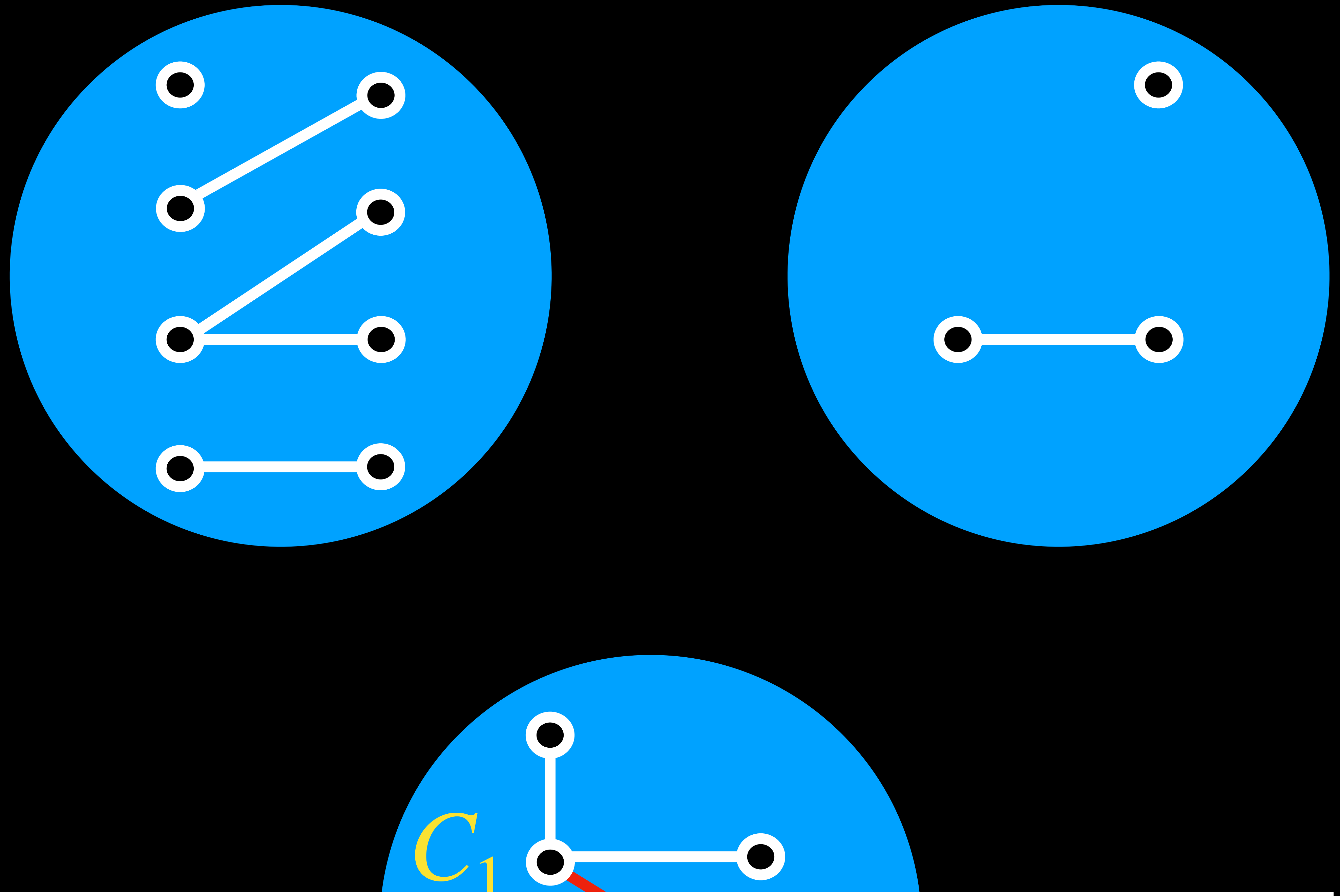
- We build an ILP based on the server capacities and component sizes
- The ILP solution reveals how we shall assign the components to the servers
- We re-solve the ILP after each edge insertion
- *Sensitivity Analysis:*
  If components of sizes $|C_1|$ and $|C_2|$ get connected, the ILP re-assigns components of size at most $O(|C_1| + |C_2|)$
- Are we done?



$$
\begin{array}{llll}
\min & \sum_{(r,m):\, r \not\geq m} x_{(r,m)} & & \\
\text{s.t.} & \sum_{(r,m)} x_{(r,m)} r_i/\delta & \geq V_i/\delta & \text{for all } i \\
 & \sum_r x_{(r,m)} & = Z_m & \text{for all } m
\end{array}
$$

Is Sensitivity Analysis Enough?

- Unfortunately, no
- We identify two different types of components:
  - *easy* components are "costly" for OPT, OPT has to pay $\Omega(|C|)$
  - for *hard* components, OPT might have no cost at all
- Sensitivity analysis works for easy components, but it is too costly for hard components
- We can afford to move easy components, *but we have to avoid moving hard components*



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How to Deal With Hard Components?

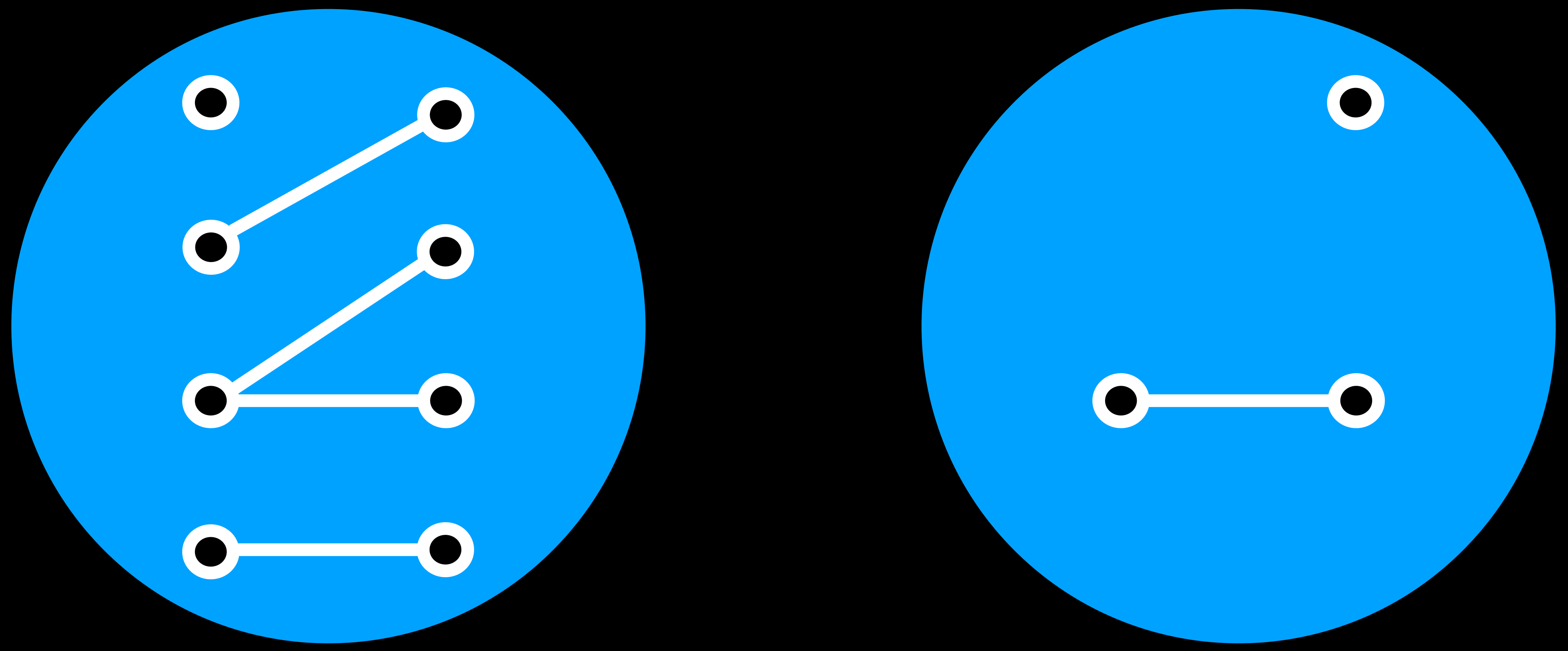
- **First approach:**
  Can we make sure we never move hard components?

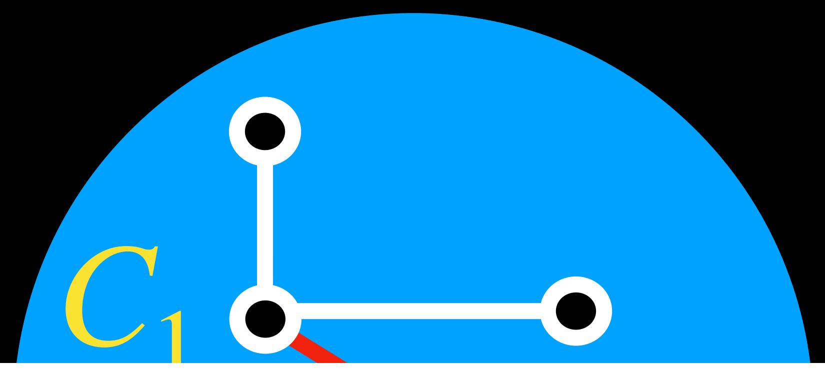
  ➡ No. Our lower bounds show that sometimes we must move hard components

- **Our approach:**
  Interleave (standard) ILP solving and *manual ILP solving*

  ➡ If the merged component is easy, use the standard ILP and sensitivity analysis

  ➡ If the merged component is hard, manually maintain an optimal ILP solution *without moving any components*





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Summary

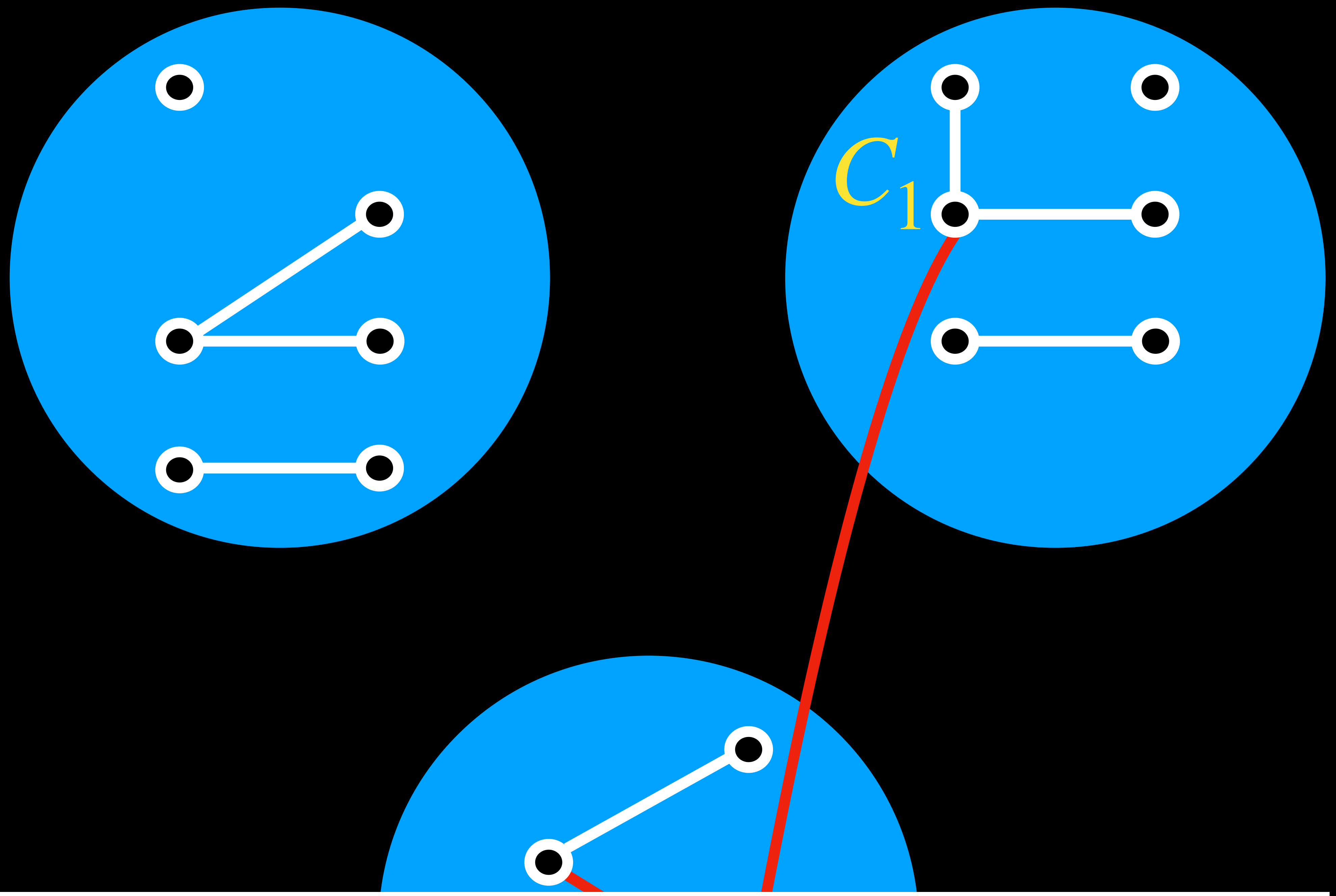
Tight Bounds for Online Graph Partitioning

- *Online graph partitioning:*
  Store n vertices across ℓ servers
  while edges are inserted into the graph and connected components must be placed on the same server

- Tight randomized algorithm,
  $\Theta(\log \ell + \log k)$-competitive

- Tight deterministic algorithm,
  $\Theta(\ell \cdot \log k)$-competitive

- *Open Problem:*
  Remove exponential dependency on ε in competitive ratio



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